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Triangular $D(-1)$-tuples

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Abstract

In this paper, we consider the extendibility of the triangular $D(-1)$-tuples, i.e., the sets of the positive integers with the property that the product of any two of them decreased by 1 is the triangular number. We prove that the only triangular $D(-1)$-triples of the form $\{1, 2, c\}$, $c = 2^n p$, where n is a non-negative integer and p is a prime, are those with $c \in \{11, 46, 352, 11936\}$. In addition, we prove that for these c's no triangular $D(-1)$-quadruple of the form $\{1, 2, c, d\}$ exists.

Key Words: Diophantine m-tuple, Pellian equation, quadratic Diophantine equation, triangular number.
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1 Introduction

Let n be an integer. A set of m positive integers $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + n$ is a square for all $1 \leq i < j \leq m$ is called a Diophantine m-tuple with the property $D(n)$ (or just a $D(n)$-m-tuple). In [4], Dujella provided a summary on past and present results on this topic, as well on its various generalizations.

The most studied case is that one with $n = 1$, where the non-existence of a $D(1)$-quintuple was proved by He, Togbé, and Ziegler in [8]. In the case of $n = -1$, Bonciocat, Cipu, and Mignotte [3] established the non-existence of $D(-1)$-quadruples. There are lots of generalizations of the problem of Diophantus in different number fields. For some details, one can consider [5], where the papers with the results from that field of research are regularly arranged chronologically.

Recently, in [6], Filipin and Szalay considered one more variation of the problem of Diophantus. Namely, they considered a set $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + 1$ is a triangular number, i.e., a number of the form $T_k = \frac{k(k+1)}{2}$, where k is a positive integer. They named such a set triangular Diophantine m-tuple and proved that there does not exist a triangular Diophantine quadruple of the form $\{1, 2, c, d\}$, with integers c and d.

The generalization of the concept of triangular Diophantine m-tuple is given through the following definition:

Definition 1.1. *Let n be an integer and m a positive integer. A set $\{a_1, \ldots, a_m\}$ of m distinct positive integers is called triangular Diophantine m-tuple (or triangular $D(n)$-m-tuple) if $a_i a_j + n$ is a triangular number for $1 \leq i < j \leq m$.*

If $\{a_1, \ldots, a_m\}$ is a triangular $D(n)$-m-tuple it follows that there exist positive integers x_{ij} such that

$$a_i a_j + n = \frac{x_{ij}(x_{ij}+1)}{2},\ 1 \le i < j \le m.$$

From these equations we obtain

$$8a_i a_j + 8n + 1 = (2x_{ij}+1)^2,\ 1 \le i < j \le m.$$

Like in the case of Diophantine m-tuples, we may consider the elliptic curve induced by the triangular $D(-1)$-triple $\{a_1, a_2, a_3\}$:

$$y^2 = (8a_1 x + 8n + 1)(8a_2 x + 8n + 1)(8a_3 x + 8n + 1).$$

According to the Siegel theorem (see [13]), the number of integer points on an elliptic curve is finite. This leads to the conclusion that there does not exist an infinite triangular $D(n)$-tuple.

In this paper, we consider the case where $n = -1$. More precisely, we investigate the extendibility of the triangular $D(-1)$-pair $\{1, 2\}$ which consists of two smallest possible elements. The main results are given as follows:

Theorem 1.2. *Let n be a non-negative integer, p a prime and $c = 2^n p$. If $\{1, 2, c\}$ is a triangular $D(-1)$-triple, then*

$$(n, p) \in \{(0, 11), (1, 23), (5, 11), (5, 373)\},$$

i.e.,

$$c \in \{11, 46, 352, 11936\}.$$

Theorem 1.3. *If $c \in \{11, 46, 352, 11936\}$, then there does not exist a triangular $D(-1)$-quadruple of the form $\{1, 2, c, d\}$.*

Note that the case $c = 11$ is the only possible extension of the triangular $D(-1)$-pair $\{1, 2\}$ to a triple with a prime number. As we shall see, that is also the minimal possible extension. Moreover, the minimal possible extension with an even c is for $c = 46$.

2 Triangular $D(-1)$-triples of the form $\{1, 2, c\}$

If the set $\{1, 2, c\}$, $c = 2^n p$, with a non-negative integer n and a prime p is a triangular $D(-1)$-triple, then there exist integers $x_c \ge 2, y_c \ge 3$ such that

$$\begin{aligned} c - 1 &= \frac{x_c(x_c+1)}{2}, \\ 2c - 1 &= \frac{y_c(y_c+1)}{2}. \end{aligned} \tag{1}$$

By substituting $x = 2x_c + 1, y = 2y_c + 1$, from (1) we obtain the equivalent system of equations

$$\begin{aligned} 8c - 7 &= x^2, \\ 16c - 7 &= y^2, \end{aligned} \tag{2}$$

which leads us to the equation

$$y^2 - x^2 = 8c, \tag{3}$$

i.e.,

$$y^2 - x^2 = 2^{n+3}p. \tag{4}$$

Since x, y are odd integers, the possible cases are:

$$(y - x, y + x) \in \{(2^k, 2^{n-k+3}p), (2^{n-k+3}p, 2^k) : k \in \mathbb{Z}, 0 < k \leq n + 3\}.$$

These cases together with (1) lead to the following equations:

$$-2^{2n-2k+4}p^2 + 3 \cdot 2^{n+2}p - 2^{2k-2} - 7 = 0, \quad 0 < k \leq n + 3,$$

and we obtain

$$p = 2^{2k-2n-3}\left(3 \cdot 2^n \pm 2\sqrt{2^{2n+1} - 7 \cdot 2^{2n-2k}}\right). \tag{5}$$

Since $2^{2n+1} - 7 \cdot 2^{2n-2k} = 2^{2n-2k}(2^{2k+1} - 7)$, it is easy to conclude that p is a positive integer if and only if $2^{2k+1} - 7$ is a perfect square. Thus we have to solve the Diophantine equation

$$2^{2k+1} - 7 = m^2. \tag{6}$$

This is the Ramanujan-Nagell equation, the solutions of which were conjectured by Ramanujan and proved by Nagell, and which are given by the following theorem:

Theorem 2.1 (see [7, 11]). *The positive integer solutions of the equation*

$$x^2 + 7 = 2^n$$

are $x = 1, 3, 5, 11, 181$, *corresponding to* $n = 3, 4, 5, 7, 15$.

For more details about different types, as well as some unsolved problems of the generalized Lebesgue-Ramanujan-Nagell equations, readers are referred to e.g. [2, 9, 10].

From Theorem 2.1 it follows that the positive solutions to (6) are

$$(k, m) \in \{(1, 1), (2, 5), (3, 11), (7, 181)\}.$$

Now we use those k in (5) and they generate the following possibilities for a prime p and a non-negative integer n:

$$(k, n, p) \in \{(1, 0, 2), (2, 0, 11), (3, 0, 2), (3, 1, 23), (7, 5, 11), (7, 5, 373)\}.$$

We conclude that the possible c which extends the triangular $D(-1)$-pair to triple is $c \in \{11, 46, 352, 11936\}$. Thus, we have completed the proof of Theorem 1.2.

Remark 2.2. *Note that from* (2) *we get the Pellian equation* $y^2-2x^2=7$*, whose solutions in positive integers give rise to the following possibilities for c:*

$$c_n^{1,2}=\frac{1}{64}\left((11\pm 6\sqrt{2})(17+12\sqrt{2})^n+(11\mp 6\sqrt{2})(17-12\sqrt{2})^n+42\right),\ n\geq 1. \tag{7}$$

Therefore, there are infinitely many triangular $D(-1)$*-triples of the form* $\{1,2,c\}$*. The minimal extension of the triangular* $D(-1)$*-pair* $\{1,2\}$ *to a triple is that with* $c=c_1^1=11$*, and according to Theorem 1.2 this is the only extension with a prime c. It is easy to see that the extensions from Theorem 1.2 are given by* $c_2^2=46, c_2^1=352, c_3^1=11936$ *and the minimal possible extension with an even c is for* $c=c_2^2=46$*.*

3 Triangular $D(-1)$-quadruples of the form $\{1,2,c,d\}$

In this section, we consider the extendibility of the triangular $D(-1)$-triples from Section 2 to quadruples. With the following considerations, we will arrive to the proof of Theorem 1.3.

Let $c\in\{11,46,352,11936\}$. If $\{1,2,c,d\}$ is a triangular $D(-1)$-quadruple, then there exist integers $x_d\geq 5, y_d\geq 7, z_d\geq 16$ such that the following holds:

$$d-1=\frac{x_d(x_d+1)}{2},$$
$$2d-1=\frac{y_d(y_d+1)}{2},$$
$$cd-1=\frac{z_d(z_d+1)}{2}.$$

Setting $X=2x_d+1, Y=2y_d+1, Z=2z_d+1$, we obtain

$$8d-7=X^2,$$
$$16d-7=Y^2,$$
$$8cd-7=Z^2.$$

By eliminating d we get the system of Pellian equations

$$Z^2-cX^2=7(c-1),$$
$$2Z^2-cY^2=7\left(c-2\right),$$

which is equivalent to

$$Z^2-cX^2=7(c-1), \tag{8}$$
$$(c-2)X^2-(c-1)Y^2+Z^2=0. \tag{9}$$

Now we have to consider the above system of Pellian equations. We will find the general integer solution to (9) and combine that with (8). In that point we use the following result due to Alekseyev [1]:

Theorem 3.1 (see [1, Theorem 5.]). *Let A, B, C be non-zero integers and let (x_0, y_0, z_0) be a particular non-trivial integer solution of the Diophantine equation $Ax^2 + By^2 + Cz^2 = 0$ with $z_0 \neq 0$. The general integer solution to the above equation is given by*

$$(x, y, z) = \frac{p}{q}(P_x(m, n), P_y(m, n), P_z(m, n)),$$

where m, n as well as p, q are coprime integers with $q > 0$ dividing $2\,\mathrm{lcm}(A, B)Cz_0^2$, and

$$\begin{aligned} P_x(m, n) &= x_0Am^2 + 2y_0Bmn - x_0Bn^2, \\ P_y(m, n) &= -y_0Am^2 + 2x_0Amn + y_0Bn^2, \\ P_z(m, n) &= z_0Am^2 + z_0Bn^2. \end{aligned}$$

A particular solution to (9) is $(X_0, Y_0, Z_0) = (1, 1, 1)$. Thus the above theorem implies that the general integer solution to (9) is given by

$$(X, Y, Z) = \frac{p}{q}\left(P_X(m, n), P_Y(m, n), P_Z(m, n)\right),$$

where m, n as well as p, q are coprime integers with $q > 0$, $q \mid 2\,\mathrm{lcm}(c-2, c-1)$, and

$$\begin{aligned} P_X(m, n) &= (c-2)m^2 - 2(c-1)mn + (c-1)n^2, \\ P_Y(m, n) &= -(c-2)m^2 + 2(c-2)mn - (c-1)n^2, \\ P_Z(m, n) &= (c-2)m^2 - (c-1)n^2. \end{aligned}$$

Puting those solutions into (8) we obtain

$$\begin{aligned} -\frac{(c-1)p^2}{q^2} \cdot ((c-2)^2m^4 - (4c^2 - 8c)m^3n + (6c^2 - 6c - 4)m^2n^2 \\ - (4c^2 - 4c)mn^3 + (c-1)^2n^4) = 7(c-1), \end{aligned}$$

i.e.,

$$\begin{aligned} (c-2)^2m^4 - (4c^2 - 8c)m^3n + (6c^2 - 6c - 4)m^2n^2 \\ - (4c^2 - 4c)mn^3 + (c-1)^2n^4 = -7\frac{q^2}{p^2}. \end{aligned} \tag{10}$$

Since $\gcd(p, q) = 1$ and $q \mid 2\,\mathrm{lcm}(c-2, c-1)$, from (10) it follows that we have to consider the following equation

$$\begin{aligned} (c-2)^2m^4 - (4c^2 - 8c)m^3n + (6c^2 - 6c - 4)m^2n^2 \\ - (4c^2 - 4c)mn^3 + (c-1)^2n^4 = -7q^2, \end{aligned} \tag{11}$$

for the corresponding finitely many possibilities of q. To find an appropriate integer solution m, n, with $\gcd(m, n) = 1$ we used PARI/GP [12] and conclude the following:

a) Let $c = 11$. Then we get that $q \mid 180$. The polynomial on the left-hand side in (11) is irreducible (for $\gcd(m, n) = 1$) and that leads to 18 Thue equations. For $(q, \pm m, \pm n) \in$

$\{(6,2,3),(6,4,3),(15,5,3),(15,5,6)\}$ we obtain $d=1$. If $(q,\pm m,\pm n)\in\{(2,2,1),(5,5,4),(18,8,9),(45,5,9)\}$, then $d=2$, while $(q,\pm m,\pm n)\in\{(1,1,2),(9,11,9),(10,10,11),(90,20,9)\}$ yields $d=11$.

b) If $c=46$, we have that $q\mid 3960$. Here we have to consider 48 Thue equations of the form (11). We obtain the results as follows: for $(q,\pm m,\pm n)\in\{(20,5,4),(36,9,8),(55,10,11),(99,9,11)\}$ we get $d=1$, while in the case $(q,\pm m,\pm n)\in\{(12,3,4),(33,12,11),(60,15,16),(165,15,11)\}$ it follows that $d=2$.

c) Suppose that $c=352$. Then $q\mid 245700$ and we considered 144 Thue equations (11). If $(q,\pm m,\pm n)\in\{(182,13,14),(325,26,25),(378,27,28),(675,27,25),(1638,39,42)\}$, then $d=1$. For $(q,\pm m,\pm n)\in\{(90,9,10),(315,36,35),(390,39,40),(1365,39,35)\}$ we obtain that $d=2$.

d) Let us consider $c=11936$. In this case, $q\mid 284864580$ and there are 768 Thue equations (11) to consider. We get the following results: $(q,\pm m,\pm n)\in\{(6006,77,78),(11781,154,153),(12090,155,156),(23715,155,153)\}$ yields $d=1$ and $(q,\pm m,\pm n)\in\{(2970,55,54),(11718,217,216),(12155,220,221),(47957,217,221)\}$ implies $d=2$.

In that way we have finished the proof of Theorem 1.3.

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References

[1] M. A. Alekseyev, On the intersections of Fibonacci, Pell, and Lucas Numbers, *Integers* **11** (2011), 239–259.

[2] M. A. Bennett, S. Siksek, Differences between perfect powers: the Lebesgue-Nagell equation, *Trans. Amer. Math. Soc.* **376** (2023), 335–370.

[3] N. C. Bonciocat, M. Cipu, M. Mignotte, There is no Diophantine $D(-1)$-quadruple, *J. London Math. Soc.* **105** (2022), 63–99.

[4] A. Dujella, *Diophantine m-tuples and elliptic curves*, Springer, Cham (2024).

[5] A. Dujella, *Diophantine m-tuples*, `http://web.math.pmf.unizg.hr/~duje/dtuples.html`.

[6] A. Filipin, L. Szalay, Triangular Diophantine tuples from $\{1,2\}$, *Rad HAZU* **27** (2023), 55–70.

[7] H. Hasse, Über eine diophantische Gleichung von Ramanujan-Nagell und ihre Verallgemeinerung, *Nagoya Math. J.* **27** (1966), 77–102.

[8] B. He, A. Togbé, V. Ziegler, There is no Diophantine quintuple, *Trans. Amer. Math. Soc.* **371** (2019), 6665–6709.

[9] A. Hoque, On a class of Lebesgue-Ramanujan-Nagell equations, *Period. Math. Hung.* **88** (2024), 418–428.

[10] M. Le, G. Soydan, A brief survey on the generalized Lebesgue-Ramanujan-Nagell equation, *Surv. Math. Appl.* **15** (2020), 473–523.

[11] L. J. Mordell, *Diophantine equations*, Academic Press (1969).

[12] PARI2, *The PARI Group, PARI/GP version* 2.15.4, Univ. Bordeaux, `http://pari.math.u-bordeaux.fr/` (2023).

[13] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abh. Preuss. Akad. Wiss. (1929).

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