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On certain labelings of disjoint unions of cycles and stars

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Abstract

First, an edge-magic labeling of a graph G is defined to be a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(u) + f(v) + f(uv)$ is a constant for each $uv \in E(G)$. Also, an edge-magic labeling f of G with the additional property that $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$ is called a super edge-magic labeling. In this paper, we investigate the (super) edge-magic properties of disjoint unions of cycles and stars. The super edge-magic results obtained in this paper yield felicitous labelings for the same class of graphs. We also propose a conjecture and several open problems arising from this study. Furthermore, we settle an old problem proposed by Wallis in his book "Magic Graphs".

Key Words: (Super) edge-magic graph (labeling), consecutively super edge-magic graph (labeling), felicitous graph (labeling), cycle, path, star, graph labeling.
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1 Introduction

Undefined graph theoretical notation and terminology can be found in [5] or [27]. The *vertex set* of a graph G is denoted by $V(G)$, while the *edge set* of G is denoted by $E(G)$. The cardinality of the vertex set of a graph G is called the *order* of G, while the cardinality of the edge set is the *size* of G. The *cycle* and *path* of order n are denoted by C_n and P_n, respectively. The *star* of size n is denoted by S_n.

For two graphs G_1 and G_2 with disjoint vertex sets, the *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

For the sake of brevity, we will use the notation $[a, b]$ for the interval of integers x such that $a \leq x \leq b$. On the other hand, if $a > b$, we treat $[a, b]$ as the empty set. If such a situation occurs in particular formulas for a given vertex labeling, then we ignore the corresponding portions of the formulas.

For a vertex labeling f of a graph G and nonempty subset S of $V(G)$, we will denote the set of vertex labels $\{f(v) \,|v \in S\}$ by writing $f(S)$. We will also utilize the notation: if $A \subseteq \mathbb{Z}$ and $b \in \mathbb{Z}$, then $A + b = \{a + b \mid a \in A\}$, where $\mathbb{Z}$ denotes the set of integers.

Kotzig and Rosa [20] initiated the study of magic valuations in 1970. Ringel and Llado [24] rediscovered this concept in 1996 and called this labeling edge-magic. This has become a popular term since then. A graph G is called *edge-magic* if there exists a bijective function $f : V(G) \cup E(G) \to [1, |V(G)| + |E(G)|]$ such that $f(u) + f(v) + f(uv)$ is a constant (called the *valence*) for each $uv \in E(G)$. Such a function is called an *edge-magic labeling*. For an edge-magic labeling f of a graph G, the valence k is given by the formula:

$$k = \frac{\sum_{v \in V(G)} \deg_G(v) f(v) + \sum_{e \in E(G)} f(e)}{|E(G)|},$$

where $\deg_G(v)$ denotes the degree of a vertex v in G. Some authors use the term "edge-magic total labelings" for this terminology as introduced by Wallis [26].

In 1998 Enomoto, Llado, Nakamigawa, and Ringel [6] introduced a particular type of edge-magic labelings called super edge-magic labelings. They defined an edge-magic labeling f of a graph G with the additional property that $f(V(G)) = [1, |V(G)|]$ to be a *super edge-magic labeling*. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling.

Some authors use "super edge-magic total labelings" and "super edge-magic total graphs" for super edge-magic labelings and super edge-magic graphs, respectively. In [10] Hegde and Shetty showed that the concepts of super edge-magic graphs and strongly indexable graphs (see [1] for the definition of a strongly indexable graph) are equivalent.

The following lemma found in [7] provides a necessary and sufficient condition for a graph to be super edge-magic. In light of this result, it suffices to exhibit the vertex labeling of a super edge-magic graph. However, we will also provide the valences to increase the clarity of our results.

Lemma 1. *A graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \dots, |V(G)|\}$ such that the set*

$$S = \{f(u) + f(v) \,|uv \in E(G)\}$$

consists of $|E(G)|$ consecutive integers. In such a case, f extends to a super edge-magic labeling of G with valence $k = |V(G)| + |E(G)| + s$, where $s = \min(S)$ and

$$S = [k - (|V(G)| + |E(G)|), k - (|V(G)| + 1)].$$

The study of super edge-magic labelings of graphs has proven to be a successful approach to many other problems in the last two decades since many relations with other types of labelings have been found (see [7] and [8] for instance), and relations with other concepts such as Skolem and Langford sequences (see [14]), liftings of graphs (see [15]), dual shuffle primes and Jacobsthal sequence (see [16] and [18]).

For a comprehensive survey of graph labeling problems, see Gallian [9]. For more information on super edge-magic graphs and related topics, see the books by Bača and Miller [2], Chartrand, Egan, and Zhang [4], López and Muntaner-Batle [15], and Marr and Wallis [21].

In [26] Wallis proved that $C_3 \cup S_1$ is not edge-magic, that is, super edge-magic neither. He also raised the question of which $C_m \cup S_1$ is edge-magic. This was partially answered by Park, Choi, and Bae [23] who showed that $C_m \cup S_1$ is super edge-magic if m is even and $m \neq 10$. Afterwards, the following result was established in [12] in terms of the edge dependent characteristic (see [11] for the definition of an edge dependent characteristic). To avoid ambiguity, note that $C_{10} \cup S_1$ has a super edge-magic labeling with valence 31. To see this, label the vertices of C_{10} with $2-6-3-7-4-11-5-9-8-10-2$ and the vertices of S_1 with $1-12$.

Theorem 1. *For even integers $m \geq 4$, the graph $C_m \cup S_1$ is super edge-magic.*

We end the introduction with a summary of what is accomplished in this study. In Section 2, we extend Theorem 1 introducing new results dealing with the disjoint unions of cycles and stars. Motivated by these results we also consider the edge-magic properties of the same classes of graphs in Section 3. Section 4 is devoted to providing an additional result, which is a continuation of the work conducted in Section 2. We believe that the results developed in this paper motivate new problems and a conjecture, some of which, are suggested in the subsequent pages.

2 Results on super edge-magic properties

In this section, we study the super edge-magic properties of disjoint unions of cycles and stars. We now present the following result.

Theorem 2. *The graph $C_m \cup S_1$ is super edge-magic if and only if m is even and $m \geq 4$.*

Proof. In light of Theorem 1, it remains to prove the necessity. For this reason, let $m = 2k+1$, where k is a positive integer, and suppose, to the contrary, that $G = C_{2k+1} \cup S_1$ has a super edge-magic labeling f. For each $uv \in E(G)$, define a function $g : E(G) \to \mathbb{N}$ by $g(uv) = f(u) + f(v)$, where $\mathbb{N}$ denotes the set of natural numbers. Since $|E(G)| = 2k+2$, it follows from the basic properties of super edge-magic labeling that

$$\sum_{uv \in E(G)} g(uv) = \sum_{i=1}^{2k+2} (s+i-1),$$

where $s = \min\{f(u)+f(v) \,|uv \in E(G)\}$. In this sum, each vertex label is counted twice except for the labels of the vertices of S_1, and $|V(G)| = 2k+3$. Thus, if we let $f(V(S_1)) = \{\alpha, \beta\}$, then we have

$$2\sum_{i=1}^{2k+3} i - (\alpha+\beta) = \sum_{i=1}^{2k+2} (s+i-1),$$

or, equivalently, $2k^2+11k+11 = (2k+2)s + (\alpha+\beta)$. Since $2k^2+11k+11$ can be expressed as $(2k+2)(k+4)+k+3$, it follows that $s \leq k+4$. We now consider three cases, depending on the possible values for the minimum induced edge sum s.

Case 1. Assume that $s = k+4$. Then

$$\begin{aligned}(2k+2)s + (\alpha+\beta) &= 2k^2+11k+11 \\ &= (2k+2)(k+4)+k+3.\end{aligned}$$

It follows that $\alpha+\beta = k+3$. Notice now that

$$\alpha+\beta \in \{f(u)+f(v) \,|uv \in E(G)\} = [s, s+2k+1].$$

However, since $s = k+4$, it follows by the assumption that

$$\alpha+\beta = k+3 < s = k+4,$$

which is impossible.
Case 2. Assume that $s = k + 3$. Then

$$\begin{aligned}(2k+2)\,s + (\alpha+\beta) &= 2k^2 + 11k + 11\\ &= (2k+2)\,(k+4) + k + 3\\ &= (2k+2)\,((k+3)+1) + k + 3\\ &= (2k+2)\,(k+3) + 3k + 5.\end{aligned}$$

It follows that $\alpha + \beta = 3k + 5$. Notice now that

$$\alpha + \beta \in \{f\,(u) + f\,(v)\,|uv \in E\,(G)\} = [s, s + 2k + 1].$$

However, since $s = k + 3$, it follows by the assumption that

$$\begin{aligned}\alpha + \beta = 3k + 5 &> s + 2k + 1\\ &= (k+3) + 2k + 1 = 3k + 4,\end{aligned}$$

which is impossible.
Case 3. Assume that $s \leq k + 2$. Then

$$\begin{aligned}(2k+2)\,s + (\alpha+\beta) &= 2k^2 + 11k + 11\\ &= (2k+2)\,((k+4-r)) + r\,(2k+2) + (k+3)\\ &= (2k+2)\,(k+4) + k + 3\\ &= (2k+2)\,(k+4-r) + (2r+1)\,k + 3 + 2r,\end{aligned}$$

where $r \geq 2$. Notice now that $\alpha, \beta \in \{f\,(v)\,|v \in V\,(G)\} = [1, 2k+3]$, which implies that

$$\max\{f\,(u) + f\,(v)\,|uv \in E\,(G)\} = (2k+2) + (2k+3) = 4k + 5.$$

However,

$$\alpha + \beta = (2r+1)\,k + 3 + 2r \geq 5k + 7 > 4k + 5$$

for $r \geq 2$, which is impossible. □

Next, we introduce some additional definitions and technical lemmas. In 2001 Muntaner-Batle [19] introduced the concept of special super edge-magic labeling of a bipartite graph. Let G be a bipartite graph with the two partite sets X and Y. If G has a super edge-magic labeling f with the property that $f\,(X) = [1, |X|]$ and $f\,(Y) = [|X| + 1, |V\,(G)\,|]$, then f is called a *special super edge-magic labeling*. Cattell [3] and Oshima [22] subsequently called such labelings *α-magic* and *consecutively super edge-magic*, respectively. In this paper, we prefer to use the latter terminology to emphasize the property that consecutively super edge-magic labeling uses consecutive integers in each partite set. We also refer to a bipartite graph with consecutively super edge-magic labeling as a *consecutively super edge-magic graph*.

The following lemma discovered by Cattell [3] plays an important role in establishing our next result.

Lemma 2. *Let v be any vertex of the path P_n. Then there exists a consecutively super edge-magic labeling f of P_n such that $f(v) = i$ for any $i \in [1, n]$ except in the following cases:*

1. *If v is an end-vertex of P_{4s+1}, then v cannot have label s or $3s$.*
2. *If v is a vertex adjacent to an end-vertex in P_6, then v cannot have label 2 or 5.*
3. *If v is the central vertex of P_5, then v cannot have label 1 or 5.*
4. *If P_n has odd order and v is a vertex in the smaller two partite sets, then v cannot have label $(n+1)/2$.*

In [17] López, Muntaner-Batle, and Rius-Font found the following fact while studying the perfect super edge-magic properties of paths.

Lemma 3. *For any positive integer n, the valence of any super edge-magic labeling of P_{2n} is exactly $5n + 1$.*

With the aid of Lemmas 2 and 3, we show the following lemma, which will be useful to establish our main result.

Lemma 4. *For every positive integer k, there exists a consecutively super edge-magic labeling of P_{8k} in which the end-vertices have labels 4 and $8k - 3$.*

Proof. By Lemma 2, there exists a consecutively super edge-magic labeling of P_{8k} in which an end-vertex has label 4. Also, Lemma 3 yields that the valence of any such labeling must be $20k + 1$. On the other hand, the valence of P_{8k} can be computed by

$$\frac{2\sum_{i=1}^{8k} i - (4 + x) + \sum_{i=8k+1}^{16k-1} i}{8k - 1},$$

where x denotes the label of the end-vertex not labeled 4. Thus,

$$\frac{2\sum_{i=1}^{8k} i - (4 + x) + \sum_{i=8k+1}^{16k-1} i}{8k - 1} = 20k + 1.$$

Solving the above equation for x, we obtain $x = 8k - 3$ as desired. □

It is known from [12] and [23] that $C_4 \cup S_1$ has a super edge-magic labeling with valence 17 by labeling the vertices of C_4 with $2 - 3 - 5 - 4 - 2$ and labeling the vertices of S_1 with $1 - 6$. This is the unique super edge-magic labeling of $C_4 \cup S_1$. It seems that the graph $C_4 \cup S_n$ is not super edge-magic for all integers $n \geq 2$. We have verified this by exhaustive computer search for $n \in [2, 50]$. However, with the aid of Lemma 4, it is now possible to present the following result.

Theorem 3. *If m and n are integers such that $m \geq 6$ is even and $n \geq 1$, then the graph $C_m \cup S_n$ is super edge-magic.*

Proof. In light of Theorem 2, it suffices to show the theorem for integers $n \geq 2$. To do so, let $G = C_m \cup S_n$ and define the graph G with

$$V(G) = \{x_i \,|i \in [1,m]\} \cup \{y\} \cup \{y_i \,|i \in [1,n]\}$$

and

$$E(G) = \{x_i x_{i+1} \,|i \in [1,m-1]\} \cup \{x_1 x_m\} \cup \{y y_i \,|i \in [1,n]\}.$$

For even integers $m \geq 6$, and integers $n \geq 2$, consider five cases, according to the possible values for the integer m.

Case 1. For $m = 8k-2$, where k is a positive integer, let $f : V(G) \to [1, 8k+n-1]$ be the vertex labeling such that

$$f(x_j) = \begin{cases} n-2+i & \text{if } j = 2i-1 \text{ and } i \in [1, 2k+1] \\ 4k+n+i & \text{if } j = 2i \text{ and } i \in [1, 2k-1] \\ 2k+n & \text{if } j = 4k \\ 6k+n-1+2i & \text{if } j = 4k-2+4i \text{ and } i \in [1,k] \\ 2k+n+2i & \text{if } j = 4k-1+4i \text{ and } i \in [1,k-1] \\ 6k+n-2+2i & \text{if } j = 4k+4i \text{ and } i \in [1,k-1] \\ 2k+n-1+2i & \text{if } j = 4k+1+4i \text{ and } i \in [1,k-1], \end{cases}$$

$f(y) = 4k+n-1$, $f(y_i) = i$ $(i \in [1, n-2])$, $f(y_{n-1}) = 4k+n$ and $f(y_n) = 8k+n-2$. Then f assigns all integers 1 through $8k+n-1$ exactly once to the vertices of G and has the property that

$$S = \{f(u) + f(v) \,|uv \in E(G)\} = [4k+n, 12k+2n-3].$$

Since $|E(G)| = 8k+n-2$, it follows that the set S consists of $8k+n-2$ consecutive integers. It is now immediate that

$$\begin{aligned} f(u) + f(v) + f(uv) &= |V(G)| + |E(G)| + \min(S) \\ &= (8k+n-1) + (8k+n-2) + (4k+n) \\ &= 20k+3n-3 \end{aligned}$$

for each $uv \in E(G)$. Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $20k+3n-3$.

Case 2. For $m = 8k$, where k is a positive integer, let $f : V(G) \to [1, 8k+n+1]$ be the vertex labeling such that

$$f(x_j) = \begin{cases} 4k+1+i & \text{if } j = 2i-1 \text{ and } i \in [1, 2k] \\ i & \text{if } j = 2i \text{ and } i \in [1, 2k] \\ 6k+n+1+2i & \text{if } j = 4k-3+4i \text{ and } i \in [1,k] \\ 2k+1+2i & \text{if } j = 4k-2+4i \text{ and } i \in [1,k] \\ 6k+n+2i & \text{if } j = 4k-1+4i \text{ and } i \in [1,k] \\ 2k+2i & \text{if } j = 4k+4i \text{ and } i \in [1,k], \end{cases}$$

$f(y) = 2k+1$ and $f(y_i) = 6k+1+i$ $(i \in [1,n])$. Then f assigns all integers 1 through $8k+n+1$ exactly once to the vertices of G and has the property that

$$S = \{f(u) + f(v) \,|uv \in E(G)\} = [4k+3, 12k+n+2].$$

Since $|E(G)| = 8k + n$, it follows that the set S consists of $8k + n$ consecutive integers. It is now immediate that

$$\begin{aligned} f(u) + f(v) + f(uv) &= |V(G)| + |E(G)| + \min(S) \\ &= (8k + n + 1) + (8k + n) + (4k + 3) \\ &= 20k + 2n + 4 \end{aligned}$$

for each $uv \in E(G)$. Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $20k + 2n + 4$.

Case 3. For $m = 8k + 4$, where k is a positive integer, let $f : V(G) \to [1, 8k + n + 5]$ be the vertex labeling such that

$$f(x_j) = \begin{cases} 4k + 3 + i & \text{if } j = 2i - 1 \text{ and } i \in [1, 2k + 1] \\ i & \text{if } j = 2i \text{ and } i \in [1, 2k + 1] \\ 6k + n + 4 + 2i & \text{if } j = 4k - 1 + 4i \text{ and } i \in [1, k] \\ 2k + 2 + 2i & \text{if } j = 4k + 4i \text{ and } i \in [1, k - 1] \\ 6k + n + 3 + 2i & \text{if } j = 4k + 1 + 4i \text{ and } i \in [1, k - 1] \\ 2k + 1 + 2i & \text{if } j = 4k + 2 + 4i \text{ and } i \in [1, k - 1], \end{cases}$$

$$(f(x_j))_{j=8k}^{8k+4} = (4k + 1, 8k + n + 3, 4k + 3, 8k + n + 5, 4k + 2),$$

$f(y) = 2k + 2$ and $f(y_i) = 6k + 4 + i$ ($i \in [1, n]$). Then f assigns all integers 1 through $8k + n + 5$ exactly once to the vertices of G and has the property that

$$S = \{f(u) + f(v) \,|uv \in E(G)\} = [4k + 5, 12k + n + 8].$$

Since $|E(G)| = 8k + n + 4$, it follows that the set S consists of $8k + n + 4$ consecutive integers. It is now immediate that

$$\begin{aligned} f(u) + f(v) + f(uv) &= |V(G)| + |E(G)| + \min(S) \\ &= (8k + n + 5) + (8k + n + 4) + (4k + 5) \\ &= 20k + 2n + 14 \end{aligned}$$

for each $uv \in E(G)$. Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $20k + 2n + 14$.

Case 4. For $m \equiv 10 \pmod{16}$, there are two subcases to pursue.

Subcase 4.1. For $m = 10$, let $f : V(G) \to [1, n + 11]$ be the vertex labeling such that

$$(f(x_i))_{i=1}^{10} = (1, 9, 3, 11, 8, 12, 4, 13, 5, 10),$$

$f(y) = 7$, $f(y_1) = 2$, $f(y_2) = 6$ and $f(y_i) = 11 + i$ ($i \in [3, n]$). Then f assigns all integers 1 through $n + 11$ exactly once to the vertices of G and has the property that

$$S = \{f(u) + f(v) \,|uv \in E(G)\} = [9, n + 18].$$

Since $|E(G)| = n + 10$, it follows that the set S consists of $n + 10$ consecutive integers. It is now immediate that

$$\begin{aligned} f(u) + f(v) + f(uv) &= |V(G)| + |E(G)| + \min(S) \\ &= (n + 11) + (n + 10) + 9 \\ &= 2n + 30 \end{aligned}$$

for each $uv \in E(G)$. Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $2n+30$.

Subcase 4.2. For $m = 16k+10$, where k is a positive integer, let $f : V(G) \to [1, 16k+n+11]$ be the vertex labeling such that

$$f(x_j) = \begin{cases} 8k+6-i & \text{if } j = 2i-1 \text{ and } i \in [1, 4k+2] \\ 12k+8 & \text{if } j = 8k+5 \\ 8k+7-i & \text{if } j = 2i-1 \text{ and } i \in [4k+4, 8k+4] \\ 1 & \text{if } j = 16k+9 \\ 16k+14-i & \text{if } j = 2i \text{ and } i \in [1, 4k+3] \\ 12k+11-2i & \text{if } j = 8k+4+4i \text{ and } i \in [1, 2k+1] \\ 12k+8-4i & \text{if } j = 8k+2+8i \text{ and } i \in [1, k] \\ 12k+14-4i & \text{if } j = 8k+6+8i \text{ and } i \in [1, k] \\ 8k+10 & \text{if } j = 16k+10, \end{cases}$$

$f(y) = 8k+7$, $f(y_1) = 2$, $f(y_2) = 8k+6$ and $f(y_i) = 16k+11+i$ $(i \in [3, n])$. Then f assigns all integers 1 through $16k+n+11$ exactly once to the vertices of G and has the property that

$$S = \{f(u) + f(v) \,|uv \in E(G)\} = [8k+9, 24k+n+18].$$

Since $|E(G)| = 16k+n+10$, it follows that the set S consists of $16k+n+10$ consecutive integers. It is now immediate that

$$\begin{aligned} f(u) + f(v) + f(uv) &= |V(G)| + |E(G)| + \min(S) \\ &= (16k+n+11) + (16k+n+10) + (8k+9) \\ &= 40k+2n+30 \end{aligned}$$

for each $uv \in E(G)$. Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $40k+2n+30$.

Case 5. For $m \equiv 2 \pmod{16}$, there are three subcases to pursue.

Subcase 5.1. For $m = 18$, let $f : V(G) \to [1, n+19]$ be the vertex labeling such that

$$(f(x_i))_{i=1}^{18} = (9, 21, 8, 20, 7, 19, 6, 18, 5, 14, 17, 15, 3, 12, 4, 16, 1, 13),$$

$f(y) = 11$, $f(y_1) = 2$, $f(y_2) = 10$ and $f(y_i) = 19+i$ $(i \in [3, n])$ assigns all integers 1 through $n+19$ exactly once to the vertices of G and has the property that

$$S = \{f(u) + f(v) \,|uv \in E(G)\} = [13, n+30].$$

Since $|E(G)| = n+18$, it follows that the set S consists of $n+18$ consecutive integers. It is now immediate that

$$\begin{aligned} f(u) + f(v) + f(uv) &= |V(G)| + |E(G)| + \min(S) \\ &= (n+19) + (n+18) + 13 \\ &= 2n+50 \end{aligned}$$

for each $uv \in E(G)$. Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $2n+50$.

Subcase 5.2. For $m=34$, let $f:V(G)\rightarrow[1,n+35]$ be the vertex labeling such that

$$\begin{aligned}(f(x_i))_{i=1}^{34} &= (17,37,16,36,15,35,14,34,13,33,12,32,11,31,10,30,9,\\ &\quad 27,29,26,7,28,6,25,5,24,8,20,4,23,3,22,1,21),\end{aligned}$$

$f(y)=19$, $f(y_1)=2$, $f(y_2)=18$ and $f(y_i)=35+i$ $(i\in[3,n])$ assigns all integers 1 through $n+35$ exactly once to the vertices of G and has the property that

$$S=\{f(u)+f(v)\,|uv\in E(G)\}=[21,n+54].$$

Since $|E(G)|=n+34$, it follows that the set S consists of $n+34$ consecutive integers. It is now immediate that

$$\begin{aligned}f(u)+f(v)+f(uv) &= |V(G)|+|E(G)|+\min(S)\\ &= (n+35)+(n+34)+21\\ &= 2n+90\end{aligned}$$

for each $uv\in E(G)$. Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $2n+90$.

Subcase 5.3. For $m=16k+34$, where k is a positive integer, let $f:V(G)\rightarrow[1,16k+n+35]$ be the vertex labeling such that

$$f(x_j)=\begin{cases}8k+18-i & \text{if } j=2i-1 \text{ and } i\in[1,4k+9]\\ 16k+38-i & \text{if } j=2i \text{ and } i\in[1,4k+8]\\ 12k+25+2i & \text{if } j=8k+17+i \text{ and } i\in[1,2]\\ 12k+24+2i & \text{if } j=8k+18+2i \text{ and } i\in[1,2]\\ 4k+8-i & \text{if } j=8k+19+2i \text{ and } i\in[1,3]\\ 12k+26-i & \text{if } j=8k+22+2i \text{ and } i\in[1,2]\\ 4k+8 & \text{if } j=8k+27\\ g(w_{8k+2-2i})+4k+23 & \text{if } j=8k+26+2i \text{ and } i\in[1,4k]\\ g(w_{8k+1-2i})+4 & \text{if } j=8k+27+2i \text{ and } i\in[1,4k],\end{cases}$$

where g is a consecutively super edge-magic labeling of P_{8k} with the two partite sets $X=\{w_{2i-1}\,|i\in[1,4k]\}$ and $Y=\{w_{2i}\,|i\in[1,4k]\}$ satisfying the condition $g(w_1)=4$ and $g(w_{8k})=8k-3$ as given in Lemma 4 so that $g(X)=[1,4k]$ and $g(Y)=[4k+1,8k]$,

$$(f(x_j))_{j=16k+28}^{16k+34}=(8k+20,4,8k+23,3,8k+22,1,8k+21),$$

$f(y)=8k+19$, $f(y_1)=2$, $f(y_2)=8k+18$ and $f(y_i)=16k+35+i$ $(i\in[3,n])$. Then f

has the properties that

$$\begin{aligned}
\{f(x_{2i-1})\,|i\in[1,4k+9]\} &= [4k+9,8k+17],\\
\{f(x_{2i})\,|i\in[1,4k+8]\} &= [12k+30,16k+37],\\
\{f(x_{i})\,|i\in[8k+18,8k+27]\} &= [4k+5,4k+8]\cup[12k+24,12k+29],\\
\{f(x_{8k+27+2i})\,|i\in[1,4k]\} &= \{g(w_{8k+1-2i})+4\,|i\in[1,4k]\}\\
&= g(X)+4=[1,4k]+4\\
&= [5,4k+4],\\
\{f(x_{8k+26+2i})\,|i\in[1,4k]\} &= \{g(w_{8k+2-2i})+4k+23\,|i\in[1,4k]\}\\
&= g(Y)+4k+23=[4k+1,8k]+4k+23\\
&= [8k+24,12k+23],\\
\{f(x_{i})\,|i\in[16k+28,16k+34]\} &= \{1,3,4\}\cup[8k+20,8k+23],\\
\{f(v)\,|v\in V(S_n)\} &= \{2,8k+18,8k+19\}\cup[16k+38,16k+n+35].
\end{aligned}$$

Thus, f assigns all integers 1 through $16k+n+35$ exactly once to the vertices of G. At this point, notice that Lemmas 1 and 3 yield that

$$\{g(u)+g(v)\,|uv\in E(P_{8k})\}=[4k+2,12k]\,.$$

Combining these, we obtain

$$S=\{f(u)+f(v)\,|uv\in E(G)\}=[8k+21,24k+n+56]\,.$$

Since $|E(G)|=16k+n+34$, it follows that the set S consists of $16k+n+34$ consecutive integers. It is now immediate that

$$\begin{aligned}
f(u)+f(v)+f(uv) &= |V(G)|+|E(G)|+\min(S)\\
&= (16k+n+35)+(16k+n+34)+(8k+21)\\
&= 40k+2n+90
\end{aligned}$$

for each $uv\in E(G)$. Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $40k+2n+90$.

□

Super edge-magic labelings obtained from the proof of Theorem 3 are illustrated next for some small values of the integers m and n.

- For $m=14$ and $n=2$, we have a super edge-magic labeling f of $C_{14}\cup S_2$ with valence 43:
$$(f(x_i))_{i=1}^{14}=(1,11,2,12,3,13,4,6,5,15,8,14,7,17),$$
$f(y)=9$ and $(f(y_i))_{i=1}^{2}=(10,16)$.
- For $m=14$ and $n=3$, we have a super edge-magic labeling f of $C_{14}\cup S_3$ with valence 46:
$$(f(x_i))_{i=1}^{14}=(2,12,3,13,4,14,5,7,6,16,9,15,8,18),$$
$f(y)=10$ and $(f(y_i))_{i=1}^{3}=(1,11,17)$.

- For $m = 16$ and $n = 3$, we have a super edge-magic labeling f of $C_{16} \cup S_3$ with valence 50:
$$(f(x_i))_{i=1}^{16} = (10, 1, 11, 2, 12, 3, 13, 4, 18, 7, 17, 6, 20, 9, 19, 8),$$
$f(y) = 5$ and $(f(y_i))_{i=1}^{3} = (14, 15, 16)$.

- For $m = 20$ and $n = 3$, we have a super edge-magic labeling f of $C_{20} \cup S_3$ with valence 60:
$$(f(x_i))_{i=1}^{20} = (12, 1, 13, 2, 14, 3, 15, 4, 16, 5, 21, 8, 20, 7, 23, 9, 22, 11, 24, 10),$$
$f(y) = 6$ and $(f(y_i))_{i=1}^{3} = (17, 18, 19)$.

- For $m = 26$ and $n = 2$, we have a super edge-magic labeling f of $C_{26} \cup S_2$ with valence 74:
$$\begin{aligned}(f(x_i))_{i=1}^{26} = \; & (13, 29, 12, 28, 11, 27, 10, 26, 9, 25, 8, 24, 20, \\ & 23, 7, 21, 6, 16, 5, 19, 4, 22, 3, 17, 1, 18),\end{aligned}$$
$f(y) = 15$ and $(f(y_i))_{i=1}^{2} = (2, 14)$.

- For $m = 26$ and $n = 3$, we have a super edge-magic labeling f of $C_{26} \cup S_3$ with valence 76:
$$\begin{aligned}(f(x_i))_{i=1}^{26} = \; & (13, 29, 12, 28, 11, 27, 10, 26, 9, 25, 8, 24, 20, \\ & 23, 7, 21, 6, 16, 5, 19, 4, 22, 3, 17, 1, 18),\end{aligned}$$
$f(y) = 15$ and $(f(y_i))_{i=1}^{3} = (2, 14, 30)$.

- For $m = 50$ and $n = 3$, consider a consecutively super edge-magic labeling g of P_8 such that $(g(w_i))_{i=1}^{8} = (4, 8, 3, 7, 2, 6, 1, 5)$. Then we have a super edge-magic labeling f of $C_{50} \cup S_3$ with valence 146:
$$\begin{aligned}(f(x_i))_{i=1}^{50} = \; & (25, 53, 24, 52, 23, 51, 22, 50, 21, 49, 20, 48, \\ & 19, 47, 18, 46, 17, 45, 16, 44, 15, 43, 14, 42, \\ & 13, 39, 41, 38, 11, 40, 10, 37, 9, 36, 12, 32, \\ & 5, 33, 6, 34, 7, 35, 8, 28, 4, 31, 3, 30, 1, 29),\end{aligned}$$
$f(y) = 27$ and $(f(y_i))_{i=1}^{3} = (2, 26, 54)$.

- For $m = 66$ and $n = 3$, consider a consecutively super edge-magic labeling g of P_{16} such that $(g(w_i))_{i=1}^{16} = (4, 11, 5, 12, 6, 14, 7, 15, 8, 16, 3, 10, 2, 9, 1, 13)$. Then we have a super edge-magic labeling f of $C_{66} \cup S_3$ with valence 176:
$$\begin{aligned}(f(x_i))_{i=1}^{66} = \; & (33, 69, 32, 68, 31, 67, 30, 66, 29, 65, 28, \\ & 64, 27, 63, 26, 62, 25, 61, 24, 60, 23, 59, \\ & 22, 58, 21, 57, 20, 56, 19, 55, 18, 54, 17, \\ & 51, 53, 50, 15, 52, 14, 49, 13, 48, 16, 44, \\ & 5, 40, 6, 41, 7, 47, 12, 46, 11, 45, 10, \\ & 43, 9, 42, 8, 36, 4, 39, 3, 38, 1, 37),\end{aligned}$$

$f(y) = 35$ and $(f(y_i))_{i=1}^{3} = (2, 34, 70)$.

We now turn our attention to the super edge-magic properties of $C_m \cup S_n$ for the case where m is odd.

Proposition 1. *If m and n are integers such that m is odd with $m \in [7, 19]$ and $n \geq 2$, then the graph $C_m \cup S_n$ is super edge-magic.*

Proof. Let m and n be integers such that m is odd with $m \in [7, 19]$ and $n \geq 2$, and define the graph $G = C_m \cup S_n$ as in the proof of Theorem 3. To see that G has a super edge-magic labeling f, label the vertices of C_m with $(f(x_i))_{i=1}^{m}$ by using the vertex labelings given in Table 1 and then label the vertices of S_n with

$$f(y) = (m+3)/2, f(y_1) = 2, f(y_2) = m + 2 \text{ and } f(y_i) = m + 1 + i \ (i \in [3, n]).$$

Consequently, f uses all integers 1 through $m+n+1$ exactly once on the vertices of G and has the property that

$$S = \{f(u) + f(v) \,|uv \in E(G)\} = [(m+7)/2, (3m+2n+5)/2].$$

Since $|E(G)| = m + n$, it follows that the set S consists of $m + n$ consecutive integers. It is now immediate that

$$\begin{aligned} f(u) + f(v) + f(uv) &= |V(G)| + |E(G)| + \min(S) \\ &= (m+n+1) + (m+n) + ((m+7)/2) \\ &= (5m+4n+9)/2 \end{aligned}$$

for each $uv \in E(G)$. Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $(5m + 4n + 9)/2$. □

Table 1: Vertex labelings of C_m for some values of odd m

m	$(f(x_i))_{i=1}^{m}$
7	$(1, 10, 3, 6, 4, 8, 7)$
9	$(1, 8, 10, 3, 12, 4, 7, 5, 9)$
11	$(1, 9, 3, 8, 6, 10, 11, 4, 14, 5, 12)$
13	$(1, 10, 3, 12, 5, 9, 7, 14, 4, 16, 6, 13, 11)$
15	$(1, 11, 3, 13, 4, 16, 6, 18, 5, 14, 7, 8, 10, 15, 12)$
17	$(1, 12, 3, 14, 4, 15, 6, 16, 8, 18, 5, 20, 7, 9, 11, 17, 13)$
19	$(1, 13, 3, 15, 4, 20, 5, 16, 6, 17, 9, 18, 10, 7, 22, 8, 12, 19, 14)$

From the preceding result, we suspect the following to be true.

Conjecture 1. *For every two integers $m \geq 3$ and $n \geq 2$, the graph $C_{2m+1} \cup S_n$ is super edge-magic.*

As we have mentioned earlier, $C_3 \cup S_1$ and $C_4 \cup S_n$ ($n \in [2,50]$) are not super edge-magic, whereas $C_4 \cup S_1$ is super edge-magic. Note now that we have verified by exhaustive computer search that the graphs $C_3 \cup S_n$ ($n \in [2,50]$) and $C_5 \cup S_n$ ($n \in \{2\} \cup [4,50]$) are not super edge-magic. However, $C_5 \cup S_3$ has a super edge-magic labeling with valence 25 by labeling the vertices of C_5 with $3-5-7-8-6-3$ and the center and end-vertices of S_3 with $9,1,2,4$, respectively. This together with the above results leads us to propose the following problem.

Problem 1. *Determine whether the graph $C_m \cup S_n$ is super edge-magic for integers m and n with $m \in [3,5]$ and $n \geq 2$.*

3 Results on edge-magic properties

In light of the results obtained in the preceding section, we consider the edge-magic properties of disjoint unions of cycles and stars in this section.

Theorem 4. *For every two integers $m \geq 4$ and $n \geq 1$, the graph $C_m \cup S_n$ is edge-magic.*

Proof. First, define the graph $G = C_m \cup S_n$ as in the proof of Theorem 3, and assume that m and n are integers with $m \geq 4$ and $n \geq 1$. In light of Theorems 2 and 3, it suffices to construct the edge-magic labeling $f : V(G) \cup E(G) \to [1, 2m+2n+1]$ when $m = 4$, or m is odd and $m \geq 5$. To this end, consider three cases, depending on the possible values for the integer m.

Case 1. For $m = 4$, let

$$f(w) = \begin{cases} 2i-1 & \text{if } w = x_{2i-1} \text{ and } i \in [1,2] \\ n+2+3i & \text{if } w = x_{2i} \text{ and } i \in [1,2] \\ 2 & \text{if } w = y \\ 3+i & \text{if } w = y_i \text{ and } i \in [1,n] \\ 2n+15-(f(u)+f(v)) & \text{if } w = uv \text{ and } uv \in E(G). \end{cases}$$

Then f has the properties that

$$\begin{aligned} \{f(v) \mid v \in V(C_4)\} &= \{1, 3, n+5, n+8\}, \\ \{f(v) \mid v \in V(S_n)\} &= \{2\} \cup [4, n+3], \\ \{f(uv) \mid uv \in E(C_4)\} &= \{n+4, n+6, n+7, n+9\}, \\ \{f(uv) \mid uv \in E(S_n)\} &= [n+10, 2n+9], \end{aligned}$$

and $|V(G)| + |E(G)| = 2n+9$, so all integers 1 through $2n+9$ are used exactly once and $f(u) + f(v) + f(uv) = 2n+15$ for each $uv \in E(G)$. Therefore, f is an edge-magic labeling of G with valence $2n+15$.

Case 2. For $m = 4k + 1$, where k is a positive integer, let

$$f(w) = \begin{cases} n + i & \text{if } w = x_{2i-1} \text{ and } i \in [1, k+1] \\ 2k + n + 1 + i & \text{if } w = x_{2i-1} \text{ and } i \in [k+2, 2k] \\ 4k + n + 3 & \text{if } w = x_{4k+1} \\ 2k + n + 2 + i & \text{if } w = x_{2i} \text{ and } i \in [1, k] \\ n + 1 + i & \text{if } w = x_{2i} \text{ and } i \in [k+1, 2k] \\ 2k + n + 2 & \text{if } w = y \\ i & \text{if } w = y_i \text{ and } i \in [1, n] \\ 10k + 3n + 6 - (f(u) + f(v)) & \text{if } w = uv \text{ and } uv \in E(G). \end{cases}$$

Then f has the properties that

$$\begin{aligned} &\{f(y_i) \mid i \in [1, n]\} = [1, n],\ \{f(x_{2i-1}) \mid i \in [1, k+1]\} = [n+1, k+n+1], \\ &\{f(x_{2i}) \mid i \in [k+1, 2k]\} = [k+n+2, 2k+n+1],\ f(y) = 2k+n+2, \\ &\{f(x_{2i}) \mid i \in [1, k]\} = [2k+n+3, 3k+n+2], \\ &\{f(x_{2i-1}) \mid i \in [k+2, 2k]\} = [3k+n+3, 4k+n+1], \\ &f(x_{4k}x_{4k+1}) = 4k+n+2,\ f(x_{4k+1}) = 4k+n+3, \\ &\{f(x_i x_{i+1}) \mid i \in [2k+2, 4k-1]\} = [4k+n+4, 6k+n+1], \\ &f(x_1 x_{4k+1}) = 6k+n+2,\ \{f(x_i x_{i+1}) \mid i \in [1, 2k]\} = [6k+n+3, 8k+n+2], \\ &f(x_{2k+1}x_{2k+2}) = 8k+n+3,\ \{f(yy_i) \mid i \in [1, n]\} = [8k+n+4, 8k+2n+3], \end{aligned}$$

and $|V(G)| + |E(G)| = 8k + 2n + 3$, so all integers 1 through $8k + 2n + 3$ are used exactly once and $f(u) + f(v) + f(uv) = 10k + 3n + 6$ for each $uv \in E(G)$. Therefore, f is an edge-magic labeling of G with valence $10k + 3n + 6$. An illustration is given in Figure 1 for $m = 13$ and $n = 6$.

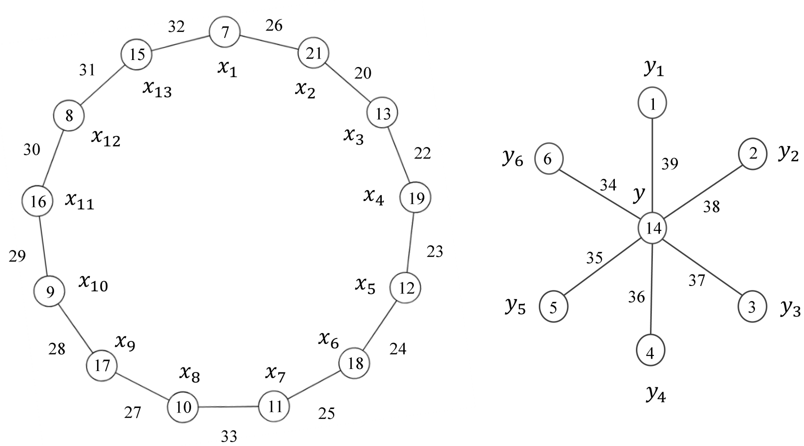

Figure 1: An edge-magic labeling of $C_{13} \cup S_6$ with valence 54

Case 3. For $m = 4k + 3$, where k is a positive integer, let

$$f(w) = \begin{cases} n+i & \text{if } w = x_{2i-1} \text{ and } i \in [1, k+2] \\ 2k+n+1+i & \text{if } w = x_{2i-1} \text{ and } i \in [k+3, 2k+2] \\ 4k+n+5 & \text{if } w = x_2 \\ 2k+n+2+i & \text{if } w = x_{2i} \text{ and } i \in [2, k] \\ k+n+3 & \text{if } w = x_{2k+2} \\ 3k+n+3 & \text{if } w = x_{2k+4} \\ n+1+i & \text{if } w = x_{2i} \text{ and } i \in [k+3, 2k+1] \\ 2k+n+3 & \text{if } w = y \\ i & \text{if } w = y_i \text{ and } i \in [1, n] \\ 10k+3n+11-(f(u)+f(v)) & \text{if } w = uv \text{ and } uv \in E(G). \end{cases}$$

Then f has the properties that

$$\begin{aligned} &\{f(y_i) \mid i \in [1,n]\} = [1,n],\ \{f(x_{2i-1}) \mid i \in [1,k+2]\} = [n+1, k+n+2], \\ &f(x_{2k+2}) = k+n+3,\ \{f(x_{2i}) \mid i \in [k+3, 2k+1]\} = [k+n+4, 2k+n+2], \\ &f(y) = 2k+n+3,\ \{f(x_{2i}) \mid i \in [2,k]\} = [2k+n+4, 3k+n+2], \\ &f(x_{2k+4}) = 3k+n+3,\ \{f(x_{2i-1}) \mid i \in [k+3, 2k+2]\} = [3k+n+4, 4k+n+3], \\ &f(x_{2k+4}x_{2k+5}) = 4k+n+4,\ f(x_2) = 4k+n+5, \\ &\{f(x_i x_{i+1}) \mid i \in [2k+5, 4k+2]\} = [4k+n+6, 6k+n+3], \\ &\{f(x_i x_{i+1}) \mid i \in [1,2]\} = [6k+n+4, 6k+n+5],\ f(x_{2k+3}x_{2k+4}) = 6k+n+6, \\ &f(x_1 x_{4k+3}) = 6k+n+7,\ \{f(x_i x_{i+1}) \mid i \in [3, 2k]\} = [6k+n+8, 8k+n+5], \\ &\{f(x_i x_{i+1}) \mid i \in [2k+1, 2k+2]\} = [8k+n+6, 8k+n+7], \\ &\{f(yy_i) \mid i \in [1,n]\} = [8k+n+8, 8k+2n+7], \end{aligned}$$

and $|V(G)| + |E(G)| = 8k + 2n + 7$, so all integers 1 through $8k + 2n + 7$ are used exactly once and $f(u) + f(v) + f(uv) = 10k + 3n + 11$ for each $uv \in E(G)$. Therefore, f is an edge-magic labeling of G with valence $10k + 3n + 11$. An illustration is given in Figure 2 for $m = 15$ and $n = 6$. □

Recall that $C_3 \cup S_1$ is not edge-magic. This together with Theorems 2 and 4 implies the following result, which settles the question above posed by Wallis [26].

Corollary 1. *The graph $C_m \cup S_1$ is edge-magic if and only if $m \neq 3$.*

The results given so far lead us to propose the following problem.

Problem 2. *Determine for which integers $n \geq 2$ is the graph $C_3 \cup S_n$ edge-magic.*

Note that we have verified by exhaustive computer search that the graph $C_3 \cup S_n$ is not edge-magic for $n \in [2, 4]$.

4 Felicitous labelings

In this section, we consider a type of graph labeling related to super edge-magic labeling. A *felicitous labeling* of a graph G is an injective function $f : V(G) \to [0, |E(G)|]$ satisfying the condition that the edge labels induced by $f(u) + f(v) \pmod{|E(G)|}$ for each $uv \in E(G)$

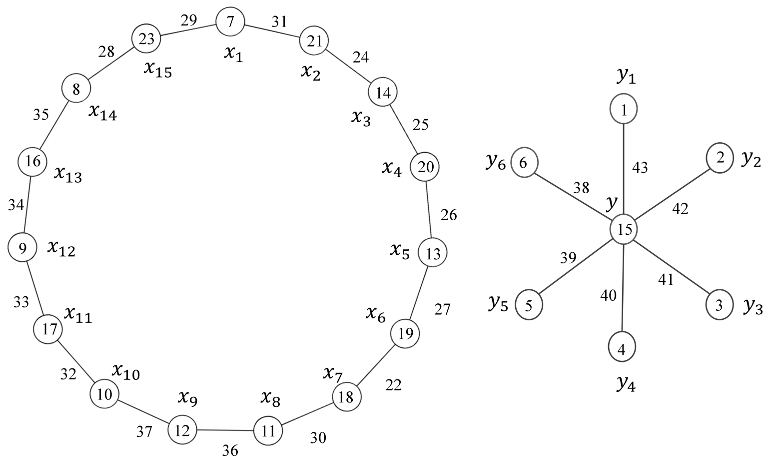

Figure 2: An edge-magic labeling of $C_{15} \cup S_6$ with valence 59

are distinct. A *felicitous graph* is a graph that admits a felicitous labeling. This definition first appeared in the paper by Shee [25]. In [13] Lee, Schmeichel, and Shee attribute to Eng Choo the concept of felicitous graphs and mention that Gary Stuart Bloom coined the terminology for these graphs.

The following relation among labelings of graphs was established in [8].

Lemma 5. *If G is a super edge-magic graph with $|E(G)| \geq |V(G)| - 1$, then G is felicitous.*

In [25] Shee has shown that the graph $C_{2m+1} \cup S_1$ is felicitous for all positive integers m. This has been strengthened by Lee, Schmeichel, and Shee [13] as stated next.

Theorem 5. *For every two positive integers m and n, the graph $C_{2m+1} \cup S_n$ is felicitous.*

The preceding result is now extended as indicated next.

Theorem 6. *For every two integers $m \geq 3$ and $n \geq 1$, the graph $C_m \cup S_n$ is felicitous unless $m = 4$ and $n \geq 3$ is odd.*

Proof. Applying Lemma 5 together with Theorems 2 and 3, we obtain a felicitous labeling of $C_{2m} \cup S_n$ for every two integers $m \geq 2$ and $n \geq 1$ unless $m = 2$ and $n \geq 3$ is odd. Thus, in light of this and Theorem 5, it remains to show that the graph $C_4 \cup S_{2k}$ has a felicitous labeling for all positive integers k. For this purpose, define the graph $G = C_m \cup S_n$ as in the proof of Theorem 3, and let $m = 4$ and $n = 2k$, where k is a positive integer. Then

$|E(G)| = 2k+4$ and the vertex labeling $f : V(G) \to [0, 2k+4]$ such that

$$f(w) = \begin{cases} 2i-2 & \text{if } w = x_i \text{ and } i \in [1,2] \\ k+i & \text{if } w = x_i \text{ and } i \in [3,4] \\ 1 & \text{if } w = y \\ 2+i & \text{if } w = y_i \text{ and } i \in [1,k] \\ 4+i & \text{if } w = y_i \text{ and } i \in [k+1,2k] \end{cases}$$

assigns all integers 0 through $2k+4$ exactly once to the vertices of G. It is now immediate that f has the properties that

$$\{f(u)+f(v) \mid uv \in E(C_4)\} = \{2, k+4, 2k+5, 2k+7\}$$

and

$$\{f(u)+f(v) \mid uv \in E(S_{2k})\} = [4, k+3] \cup [k+6, 2k+5],$$

which shows that the edge labels induced by $f(u)+f(v) \pmod{2k+4}$ for each $uv \in E(G)$ are distinct. Therefore, f is a felicitous labeling of G. □

It is now important to remark that the felicitous properties of $C_{2m} \cup S_n$ ($m \geq 2$ and $n \geq 1$) have never been examined before.

We close this section with the following problem.

Problem 3. *Determine for which positive integers n is the graph $C_4 \cup S_{2n+1}$ felicitous.*

It is clear from Theorem 2 and Lemma 5 that $C_4 \cup S_1$ is felicitous. However, it seems that the graph $C_4 \cup S_{2n+1}$ is not felicitous for all positive integers n. With the aid of a computer, we have verified this for $n \in [1, 50]$.

5 Conclusions

In this paper, we established new edge-magic, super edge-magic, and felicitous properties of $C_m \cup S_n$ for several families of parameters (m, n). In particular, we settled a question posed by Wallis concerning $C_m \cup S_1$. For some of the remaining unresolved cases, we also obtained negative evidence by exhaustive computer search in the explicit parameter ranges recorded above. These results lead naturally to the conjecture and open problems stated in the paper.

References

[1] B. D. Acharya, S. M. Hegde, Strongly indexable graphs, *Discrete Math.* **93** (1991), 123–129.

[2] M. Bača, M. Miller, *Super edge-antimagic graphs: a wealth of problems and some solutions*, Brown Walker Press, Boca Raton, FL, USA (2007).

[3] R. Cattell, Graceful labellings of paths, *Discrete Math.* **307** (2007), 3161–3176.

[4] G. Chartrand, C. Egan, P. Zhang, *How to Label a Graph*, Springer Briefs in Mathematics, Springer, Cham (2019).

[5] G. Chartrand, L. Lesniak, *Graphs & Digraphs*, 3rd ed., Chapman&Hall/CRC (1996).

[6] H. Enomoto, A. Llado, T. Nakamigawa, G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (2) (1998), 105–109.

[7] R. M. Figueroa-Centeno, R. Ichishima, F. A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.* **231** (1-3) (2001), 153–168.

[8] R. M. Figueroa-Centeno, R. Ichishima, F. A. Muntaner-Batle, A. Oshima, A magical approach to some labeling conjectures, *Discuss. Math. Graph Theory* **31** (1) (2011), 79–113.

[9] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* (2025), DS6.

[10] S. Hegde, S. Shetty, Strongly k-indexable and super edge-magic labelings are equivalent, *NITK Research Bulletin* **12** (2003), 23–28.

[11] S. M. Hegde, S. Shetty, Strongly indexable graphs and applications, *Discrete Math.* **309** (2009), 6160–6168.

[12] R. Ichishima, A. Oshima, The edge dependent characteristic of cycles, *Memoirs of the Kokushikan Univ. Inf. Sci.* **40** (2019), 10–14.

[13] S. M. Lee, E. Schmeichel, S. C. Shee, On felicitous graphs, *Discrete Math.* **93** (2-3) (1991), 201–209.

[14] S. C. López, F. A. Muntaner-Batle, Langford sequences and a product of digraphs, *European J. Combin.* **53** (2016), 86–95.

[15] S. C. López, F. A. Muntaner-Batle, *Graceful, Harmonious and Magic Type Labelings: Relations and Techniques*, Springer Briefs in Mathematics, Springer, Cham (2017).

[16] S. C. López, F. A. Muntaner-Batle, M. Prabu, On the super edge-magicness of graphs of equal order and size, arXiv:1706.00211, `https://doi.org/10.48550/arXiv.1706.00211`.

[17] S. C. López, F. A. Muntaner-Batle, M. Rius-Font, Perfect super edge-magic graphs, *Bull. Math. Soc. Sci. Math. Roumanie* **55 (103)** (2012), 199–208.

[18] S. C. López, F. A. Muntaner-Batle, M. Rius-Font, The jumping knight and other (super) edge-magic constructions, *Mediterr. J. Math.* **11** (2014), 217–235.

[19] F. A. Muntaner-Batle, Special super edge-magic labelings of bipartite graphs, *J. Combin. Math. Combin. Comput.* **39** (2001), 107–120.

[20] A. Kotzig, A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[21] A. M. Marr, W. D. Wallis, *Magic Graphs*, 2nd ed., Birkhäuser/Springer, New York (2013).

[22] A. Oshima, Consecutively super edge-magic tree with diameter 4, *Rev. Bull. Calcutta Math. Soc.* **15** (2007), 87–90.

[23] J. Y. Park, J. H. Choi, J. H. Bae, On super edge-magic labeling of some graphs, *Bull. Korean Math. Soc.* **45** (2008), 11–21.

[24] G. Ringel, A. Llado, Another tree conjecture, *Bull. Inst. Combin. Appl.* **18** (1996), 83–85.

[25] S. C. Shee, On harmonious and related graphs, *Ars Combin.* **23 A** (1987), 237–247.

[26] W. D. Wallis, *Magic Graphs*, Birkhäuser, Boston (2001).

[27] D. B. West, *Introduction to Graph Theory*, Prentice-Hall, Upper Saddle River, NJ (1996).

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