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$\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-Double cyclic codes

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Abstract

Let p be a prime number and r, s be positive integers such that $r \leq s$. This paper is concerned with $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic codes. These codes can be identified as submodules of the ring

$$\mathbb{Z}_{p^r}/ < x^{\alpha} - 1 > \times \mathbb{Z}_{p^r}/ < x^{\beta} - 1 > \times \mathbb{Z}_{p^s}/ < x^{\gamma} - 1 > \times \mathbb{Z}_{p^s}/ < x^{\eta} - 1 >,$$

where α, β, γ and η are positive integers. We determine the generator polynomials and minimal generating sets for this family of codes. Furthermore, we classify $\mathbb{Z}_p\mathbb{Z}_{p^2}$-double cyclic codes.

Key Words: Double cyclic codes, generator polynomials, minimum generating sets, dual code.
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1 Introduction

Additive codes were first introduced by Delsarte in 1973 in terms of association schemes (see [9]). According to his definition, an additive code is a subgroup of the underlying abelian group in a translation association scheme. In 2018, Borges et al. [5] studied $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. A $\mathbb{Z}_2\mathbb{Z}_4$-additive code is a subgroup of $\mathbb{Z}_2^r\mathbb{Z}_4^s$, for some positive integers r and s. If $s = 0$, then $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are just binary linear codes; and if $r = 0$, then $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are the quaternary linear codes over $\mathbb{Z}_4$. So, this class of codes contains all binary and quaternary linear codes as subclasses. Moreover, this class of codes plays an important role in a wide range of applications. For example, perfect $\mathbb{Z}_2\mathbb{Z}_4$-additive codes have been utilized in the subject of steganography [14]. In 2014, T. Abualrab et al. studied $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes and obtained the minimal generating set for these codes and some optimal bound from this family of codes [1]. After that, Borges et al. studied the generating polynomials for dual codes of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes [4].

In [6], Borges et al. introduced double cyclic codes over $\mathbb{Z}_2$ and they explored the algebraic structures of this kind of codes. Gao et al. researched double cyclic codes over $\mathbb{Z}_4$ [10]. In the literature, the authors determined the generator polynomials of these codes and their duals. Following the approaches given in [1, 6, 10], we are concerned in this paper with the algebraic structures of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic codes.

In [8], the authors studied $\mathbb{Z}_{p^r}\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes, where $\mathbb{Z}_{p^n}$ is a ring of integers modulo p^n. Both rings $\mathbb{Z}_{p^r}$ and $\mathbb{Z}_{p^s}$ are finite chain rings with the maximal ideal $< p >$. They studied the algebraic structure of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes in detail. They also studied the minimal generating sets for these codes. Also, $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}\mathbb{Z}_{p^t}$-additive cyclic codes have been defined in [12] (see also [18, 15, 16, 11]). Assume that p is a prime number and that r and s are positive integers such that $r \leq s$. The purpose of this paper is a study of the algebraic structure of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic codes. Note that the $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes can be identified as $\mathbb{Z}_{p^s}[x]$-submodules of $\frac{\mathbb{Z}_{p^r}[x]}{<x^\alpha-1>} \times \frac{\mathbb{Z}_{p^r}[x]}{<x^\beta-1>} \times \frac{\mathbb{Z}_{p^s}[x]}{<x^\gamma-1>} \times \frac{\mathbb{Z}_{p^s}[x]}{<x^\eta-1>}$, where α, β, γ and η are fixed positive integers coprime with p. Clearly, the class of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic codes contains $\mathbb{Z}_{p^r}$-double cyclic codes and $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$- additive cyclic codes as subclasses. Moreover, we study generator polynomials and minimal generating sets of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-double cyclic codes and classify this class of codes. Moreover, we provide some examples of these codes.

2 Basic properties of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic codes

We begin with the following definition.

Definition 2.1. *A subset C of $\mathbb{Z}_{p^r}^\alpha \times \mathbb{Z}_{p^r}^\beta \times \mathbb{Z}_{p^s}^\gamma \times \mathbb{Z}_{p^s}^\eta$ is called a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double additive cyclic code (or, for simplicity, $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code) if*

(i) C is a subgroup of $\mathbb{Z}_{p^r}^\alpha \times \mathbb{Z}_{p^r}^\beta \times \mathbb{Z}_{p^s}^\gamma \times \mathbb{Z}_{p^s}^\eta$, that is, C is a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double additive code, and

(ii) for any codeword

$$u = (a_0, a_1, \dots, a_{\alpha-1}; b_0, b_1 \dots, b_{\beta-1}; c_0, c_1, \dots, c_{\gamma-1}; d_0, d_1, \dots, d_{\eta-1})$$

in C, its double cyclic shift

$$T(u) = (a_{\alpha-1}, a_0, \dots, a_{\alpha-2}; b_{\beta-1}, b_0, \dots, b_{\beta-2}; c_{\gamma-1}, c_0, \dots, c_{\gamma-2}; d_{\eta-1}, d_0, \dots, d_{\eta-2})$$

is also in C.

Let $u = (a_0, a_1, \dots, a_{\alpha-1}; b_0, b_1 \dots, b_{\beta-1}; c_0, c_1, \dots, c_{\gamma-1}; d_0, d_1, \dots, d_{\eta-1})$ be a codeword in C. For a positive integer i, we denote the i-th shift of u by

$$\begin{aligned} u^{(i)} = (&a_{0-i}, a_{1-i}, \dots, a_{\alpha-1-i}; b_{0-i}, b_{1-i}, \dots, b_{\beta-1-i};\\ &c_{0-i}, c_{1-i}, \dots, c_{\gamma-1-i}; d_{0-i}, d_{1-i}, \dots, d_{\eta-1-i}),\end{aligned}$$

where the subscripts are read modulo p^r, p^r, p^s and p^s, respectively.

Now, let X, Y, Z and W be the sets of $\mathbb{Z}_{p^r}$, $\mathbb{Z}_{p^r}$, $\mathbb{Z}_{p^s}$ and $\mathbb{Z}_{p^s}$ coordinate positions, respectively. Hence $|X| = \alpha$, $|Y| = \beta$, $|Z| = \gamma$ and $|W| = \eta$. Let $C_\alpha, C_\beta, C_\gamma, C_\eta$ be the punctured code of C by deleting the coordinates out of X, Y, Z and W, respectively. Clearly, if $C \subseteq \mathbb{Z}_{p^r}^\alpha \times \mathbb{Z}_{p^s}^\beta \times \mathbb{Z}_{p^s}^\gamma \times \mathbb{Z}_{p^s}^\eta$ is a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code, then C_α (respectively, C_β, C_γ and C_η) is a cyclic code over $\mathbb{Z}_{p^r}^\alpha$ (respectively, $\mathbb{Z}_{p^r}^\beta$, $\mathbb{Z}_{p^s}^\gamma$ and $\mathbb{Z}_{p^s}^\eta$).

Since any $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code C is an additive code, it must be closed under addition and multiplication by elements in $\mathbb{Z}$ as well, that is, for any element

$$c = (a_0, a_1, \dots, a_{\alpha-1}; b_0, b_1, \dots, b_{\beta-1}; c_0, c_1, \dots, c_{\gamma-1}; d_0, d_1, \dots, d_{\eta-1}) \in C,$$

and $e \in \mathbb{Z}$, we must have

$$ec = (ea_0, ea_1, \dots, ea_{\alpha-1}; eb_0, eb_1, \dots, eb_{\beta-1}; ec_0, ec_1, \dots, ec_{\gamma-1};\\ ed_0, ed_1, \dots, ed_{\eta-1}) \in C,$$

where ea_i and eb_j are performed modulo p^r for all $i = 0, 1, \dots, \alpha-1$ and $j = 0, 1, \dots, \beta-1$, and also ec_k and ed_ℓ are performed modulo p^s for all $k = 0, 1, \dots, \gamma-1$ and $\ell = 0, 1, \dots, \eta-1$. It is evident that any $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code C is a $\mathbb{Z}$-submodule as well.

For an element $u = (a; b; c; d) \in \mathbb{Z}_{p^r}^{\alpha} \times \mathbb{Z}_{p^r}^{\beta} \times \mathbb{Z}_{p^s}^{\gamma} \times \mathbb{Z}_{p^s}^{\eta}$, we define the weight of u to be

$$wt(u) = wt_L(a) + wt_L(b) + wt_L(c) + wt_L(d),$$

where $wt_L(a)$, $wt_L(b)$, $wt_L(c)$ and $wt_L(d)$ are the Lee weights of a, b, c and d, respectively. Let $d(C)$ denote the non-zero minimum distance between the distinct pairs of codewords in C.

For any two arbitrary elements

$$u = (a_0, a_1, \dots, a_{\alpha-1}; b_0, b_1, \dots, b_{\beta-1}; c_0, c_1, \dots, c_{\gamma-1}; d_0, d_1, \dots, d_{\eta-1}) \text{ and}$$
$$v = (e_0, e_1, \dots, e_{\alpha-1}; f_0, f_1, \dots, f_{\beta-1}; g_0, g_1, \dots, g_{\gamma-1}; h_0, h_1, \dots, h_{\eta-1})$$

in $\mathbb{Z}_{p^r}^{\alpha} \times \mathbb{Z}_{p^r}^{\beta} \times \mathbb{Z}_{p^s}^{\gamma} \times \mathbb{Z}_{p^s}^{\eta}$, the inner product of u and v is defined as

$$\begin{aligned} u \,.\, v = & p^{s-r}(a_0e_0 + a_1e_1 + \cdots + a_{\alpha-1}e_{\alpha-1} + b_0f_0 + b_1f_1 + \cdots + b_{\beta-1}f_{\beta-1}) \\ & + c_0g_0 + c_1g_1 + \cdots + c_{\gamma-1}g_{\gamma-1} + d_0h_0 + d_1h_1 + \cdots + d_{\eta-1}h_{\eta-1} \mod p^s \\ = & p^{s-r}\sum_{i=0}^{\alpha-1} a_ie_i + p^{s-r}\sum_{j=0}^{\beta-1} b_jf_j + \sum_{k=0}^{\gamma-1} c_kg_k + \sum_{\ell=0}^{\eta-1} d_\ell h_\ell \mod p^s. \end{aligned}$$

Definition 2.2. *Assume that $n = p^{r-1}(\alpha+\beta) + p^{s-1}(\gamma+\eta)$, and that the map*

$$\Phi : \mathbb{Z}_{p^r}^{\alpha} \times \mathbb{Z}_{p^r}^{\beta} \times \mathbb{Z}_{p^s}^{\gamma} \times \mathbb{Z}_{p^s}^{\eta} \longrightarrow \mathbb{Z}_p^n$$

is given by

$$\Phi(x; y; z; w) = \big(\Phi_r(x_1), \dots, \Phi_r(x_\alpha), \Phi_r(y_1), \dots, \Phi_r(y_\beta), \Phi_s(z_1), \dots, \Phi_s(z_\gamma),\\ \Phi_s(w_1), \dots, \Phi_s(w_\eta)\big),$$

for all $x = (x_1, x_2, \dots, x_\alpha) \in \mathbb{Z}_{p^r}^{\alpha}$, $y = (y_1, y_2, \dots, y_\beta) \in \mathbb{Z}_{p^r}^{\beta}$, $z = (z_1, z_2, \dots, z_\gamma) \in \mathbb{Z}_{p^s}^{\gamma}$ and $w = (w_1, w_2, \dots, w_\eta) \in \mathbb{Z}_{p^s}^{\eta}$, where the classical Gray map $\Phi_j : \; Z_{p^j} \to Z_p^{p^{j-1}}$ is defined in [12].

Clearly the map Φ is one-to-one but not surjective, and so, in general, it is not bijective.

Definition 2.3. *Let C be a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code. The dual code of C, denoted by $C^\perp$, is defined by*

$$C^\perp = \left\{ v \in \mathbb{Z}_{p^r}^{\alpha}\mathbb{Z}_{p^r}^{\beta}\mathbb{Z}_{p^s}^{\gamma}\mathbb{Z}_{p^s}^{\eta} \;\middle|\; u \,.\, v = 0, \text{ for all } u \in C \right\}.$$

Furthermore, C is called self-orthogonal if $C^\perp \subseteq C$ and self-dual if $C = C^\perp$.

By using the above definition of $C^{\perp}$, in conjunction with slight modifications in the proof of Lemma 3.2 in [2], one can prove the following lemma.

Lemma 2.4. *If C is a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code, then $C^{\perp}$ is also a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code.*

We denote the ring

$$\frac{\mathbb{Z}_{p^r}[x]}{< x^{\alpha}-1 >} \times \frac{\mathbb{Z}_{p^r}[x]}{< x^{\beta}-1 >} \times \frac{\mathbb{Z}_{p^s}[x]}{< x^{\gamma}-1 >} \times \frac{\mathbb{Z}_{p^s}[x]}{< x^{\eta}-1 >}$$

by $R_{\alpha,\beta,\gamma,\eta}$. Let C be a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code. An element

$$W = (a_0, a_1, \ldots, a_{\alpha-1}; b_0, b_1, \ldots, b_{\beta-1}; c_0, c_1, \ldots, c_{\gamma-1}; d_0, d_1, \ldots, d_{\eta-1})$$

in C can be identified with a module element consisting of four polynomials

$$\begin{aligned} W(x) = \big(&a_0 + a_1x + \cdots + a_{\alpha-1}x^{\alpha-1}; b_0 + b_1x + \cdots + b_{\beta-1}x^{\beta-1};\\ &c_0 + c_1x + \cdots + c_{\gamma-1}x^{\gamma-1}; d_0 + d_1x + \cdots + d_{\eta-1}x^{\eta-1}\big) \end{aligned}$$

in $R_{\alpha,\beta,\gamma,\eta}$. This identification gives a one-to-one correspondence between the elements of C and its image in $R_{\alpha,\beta,\gamma,\eta}$.

Let $f(x) \in \mathbb{Z}_{p^s}[x]$ and $\big(a(x); b(x); c(x); d(x)\big) \in R_{\alpha,\beta,\gamma,\eta}$, and consider the following multiplication:

$$\begin{aligned} f(x) * \big(a(x); b(x); c(x); d(x)\big) = (&f(x)a(x) \bmod (x^{\alpha}-1); f(x)b(x) \bmod (x^{\beta}-1);\\ &f(x)c(x) \bmod (x^{\gamma}-1); f(x)d(x) \bmod (x^{\eta}-1)). \end{aligned}$$

The products of the right-hand side are the standard polynomial products in $\mathbb{Z}_{p^r}/\langle x^{\alpha}-1\rangle$, $\mathbb{Z}_{p^r}/\langle x^{\beta}-1\rangle$, $\mathbb{Z}_{p^s}/\langle x^{\gamma}-1\rangle$ and $\mathbb{Z}_{p^s}/\langle x^{\eta}-1\rangle$, respectively. Clearly, the ring $R_{\alpha,\beta,\gamma,\eta}$ is a $\mathbb{Z}_{p^s}[x]$-module (and also $\mathbb{Z}_{p^s}$-module) with respect to the multiplication $*$. This provides the polynomial definition of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic codes as follows.

Remark 2.5. *A subset C of $R_{\alpha,\beta,\gamma,\eta}$ is called a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code if and only if*

(i) C is a subgroup of $R_{\alpha,\beta,\gamma,\eta}$, and

(ii) for any element

$$\begin{aligned} u(x) &= \big(a(x); b(x); c(x); d(x)\big)\\ &= \big(a_0 + a_1x + \cdots + a_{\alpha-1}x^{\alpha-1}; b_0 + b_1x + \cdots + b_{\beta-1}x^{\beta-1};\\ &\quad c_0 + c_1x + \cdots + c_{\gamma-1}x^{\gamma-1}; d_0 + d_1x + \cdots + d_{\eta-1}x^{\eta-1}\big) \end{aligned}$$

in C, we have that

$$\begin{aligned} x * u(x) &= x * \big(a(x); b(x); c(x); d(x)\big)\\ &= x * \big(a_0 + a_1x + \cdots + a_{\alpha-1}x^{\alpha-1}; b_0 + b_1x + \cdots + b_{\beta-1}x^{\beta-1};\\ &\quad c_0 + c_1x + \cdots + c_{\gamma-1}x^{\gamma-1}; d_0 + d_1x + \cdots + d_{\eta-1}x^{\eta-1}\big)\\ &= \big(a_{\alpha-1} + a_0x + \cdots + a_{\alpha-2}x^{\alpha-1}; b_{\beta-1} + b_0x + \cdots + b_{\beta-2}x^{\beta-1};\\ &\quad c_{\gamma-1} + c_0x + \cdots + c_{\gamma-2}x^{\gamma-1}; d_{\eta-1} + d_0x + \cdots + d_{\eta-2}x^{\eta-1}\big) \end{aligned}$$

is an element in C.

Thus $x*u(x)$ is the image of the vector $u^{(1)}$, and so the operation of $u(x)$ by x in $R_{\alpha,\beta,\gamma,\eta}$ corresponds to a shift of u. In general $x^i * u(x) = u^{(i)}(x)$ for all positive integers i. So we have the following result.

Proposition 2.6. *A subset C of $R_{\alpha,\beta,\gamma,\eta}$ is a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code if and only if C is a $\mathbb{Z}_{p^s}[x]$-submodule of $R_{\alpha,\beta,\gamma,\eta}$.*

3 A generating set $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic codes

In this section, we study submodules of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}\mathbb{Z}_{p^s}$. In fact, we describe the generators of such submodules and give their spanning sets.

Theorem 3.1. *Let C be a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code. Then*

$$\begin{aligned} C = \Big\langle\, & \big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0; 0\big)\,, \\ & \big(\ell_1(x); f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x); 0; 0\big)\,, \\ & \big(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2q_2(x) + \cdots + p^{s-1}q_{s-1}(x); 0\big)\,, \\ & \big(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)\big)\,\Big\rangle, \end{aligned}$$

where $f_1(x), f_2(x), \ell_i(x), a_j(x), b_j(x) \in \mathbb{Z}_{p^r}[x]$, for $i = 1, \cdots, 5$, $j = 1, \cdots, r-1$, and $f_3(x), f_4(x), \ell_6(x), q_k(x), g_k(x) \in \mathbb{Z}_{p^s}[x]$, for $k = 1, \cdots, s-1$, which satisfy the conditions

$$\begin{aligned} & a_{r-1}(x)|a_{r-2}(x)|\ldots|a_1(x)|f_1(x)|(x^\alpha - 1) \bmod p^r, \\ & b_{r-1}(x)|b_{r-2}(x)|\ldots|b_1(x)|f_2(x)|(x^\beta - 1) \bmod p^r, \\ & q_{s-1}(x)|q_{s-2}(x)|\ldots|q_1(x)|f_3(x)|(x^\gamma - 1) \bmod p^s, \text{ and} \\ & g_{s-1}(x)|g_{s-2}(x)|\ldots|g_1(x)|f_4(x)|(x^\eta - 1) \bmod p^s. \end{aligned}$$

Proof. Consider the projection map

$$\Psi : C \longrightarrow \mathbb{Z}_{p^s}[x]/\langle x^\eta - 1\rangle$$

given by $\Psi(a(x); b(x); c(x); d(x)) = d(x)$ for all $(a(x); b(x); c(x); d(x)) \in C$. It is easy to see that Ψ is a $\mathbb{Z}_{p^s}[x]$-module homomorphism and its image (that is, $\Psi(C)$) is an ideal of $\mathbb{Z}_{p^s}[x]/\langle x^\eta - 1\rangle$. As $\Psi(C)$ is an ideal of $\mathbb{Z}_{p^s}[x]/\langle x^\eta - 1\rangle$ and η is an odd integer, in view of [7, Theorem 6], one can write

$$\Psi(C) = \langle f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)\rangle,$$

where $f_4(x), g_1(x), \cdots, g_{s-1}(x) \in \mathbb{Z}_{p^s}[x]$ such that

$$g_{s-1}(x)|g_{s-2}(x)|\ldots|g_1(x)|f_4(x)|(x^\eta - 1) \bmod p^s.$$

Also the kernel of Ψ is

$$\begin{aligned} \mathrm{Ker}(\Psi) = \Big\{ (a(x); b(x); c(x); 0) \in C \;\Big|\; & a(x) \in \frac{\mathbb{Z}_{p^r}[x]}{\langle x^\alpha - 1\rangle}, b(x) \in \frac{\mathbb{Z}_{p^r}[x]}{\langle x^\beta - 1\rangle}, \\ & c(x) \in \frac{\mathbb{Z}_{p^s}[x]}{\langle x^\gamma - 1\rangle} \Big\}. \end{aligned}$$

Now, we put

$$I := \{(a(x); b(x); c(x)) \in R_{\alpha,\beta,\gamma} \;\big|\; (a(x); b(x); c(x); 0) \in \mathrm{Ker}(\Psi)\},$$

where $R_{\alpha,\beta,\gamma} = \mathbb{Z}_{p^r}[x]/\langle x^\alpha - 1\rangle \times \mathbb{Z}_{p^r}[x]/\langle x^\beta - 1\rangle \times \mathbb{Z}_{p^s}[x]/\langle x^\gamma - 1\rangle$. Clearly, I is an $\mathbb{Z}_{p^r}\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic code in $\mathbb{Z}_{p^r}[x]/\langle x^\alpha - 1\rangle \times \mathbb{Z}_{p^r}[x]/\langle x^\beta - 1\rangle \times \mathbb{Z}_{p^s}[x]/\langle x^\gamma - 1\rangle$. Therefore, by [8, Theorem 4.9], the ideal I can be written in the following form

$$\begin{aligned} I = \Big\langle &\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0\big), \\ &\big(\ell_1(x); f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x); 0\big), \\ &\big(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2q_2(x) + \cdots + p^{s-1}q_{s-1}(x)\big) \Big\rangle, \end{aligned}$$

where $f_1(x), f_2(x), \ell_i(x), a_j(x), b_j(x) \in \mathbb{Z}_{p^r}[x]$, for $i = 1, \ldots, 3$, $j = 1, \ldots, r-1$, and $f_3(x), q_k(x), g_k(x) \in \mathbb{Z}_{p^s}[x]$, for $k = 1, \ldots, s-1$, such that

$$\begin{aligned} &a_{r-1}(x)|a_{r-2}(x)|\ldots|a_1(x)|f_1(x)|(x^\alpha - 1) \bmod p^r, \\ &b_{r-1}(x)|b_{r-2}(x)|\ldots|b_1(x)|f_2(x)|(x^\beta - 1) \bmod p^r \text{ and} \\ &q_{s-1}(x)|q_{s-2}(x)|\ldots|q_1(x)|f_3(x)|(x^\gamma - 1) \bmod p^s. \end{aligned}$$

Now, assume that $(c_1(x); c_2(x); c_3(x); 0)$ is an arbitrary element in $\mathrm{Ker}(\Psi)$. Then

$$(c_1(x); c_2(x); c_3(x)) \in I.$$

Thus there exist polynomials $m_1(x) \in \mathbb{Z}_{p^r}[x]/\langle x^\alpha - 1\rangle$, $m_2(x) \in \mathbb{Z}_{p^r}[x]/\langle x^\beta - 1\rangle$ and $m_3(x) \in \mathbb{Z}_{p^s}[x]/\langle x^\gamma - 1\rangle$ such that

$$\begin{aligned} (c_1(x); c_2(x); c_3(x)) = {} & m_1(x) * (f_1(x) + pa_1(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0) \\ & + m_2(x) * (\ell_1(x); f_2(x) + pb_1(x) + \cdots + p^{r-1}b_{r-1}(x); 0) \\ & + m_3(x) * (\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + \cdots + p^{s-1}q_{s-1}(x)). \end{aligned}$$

This implies that $\mathrm{Ker}(\Psi)$ is a submodule of C which is reword as follows

$$\begin{aligned} \mathrm{Ker}(\Psi) = \Big\langle &\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0; 0\big), \\ &\big(\ell_1(x); f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x); 0; 0\big), \\ &\big(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2q_2(x) + \cdots + p^{s-1}q_{s-1}(x); 0\big) \Big\rangle. \end{aligned}$$

Finally, by the First Isomorphism Theorem, we have

$$\frac{C}{\mathrm{Ker}(\Psi)} \cong \langle f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)\rangle.$$

Now, let $(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)) \in C$ be such that

$$\begin{aligned} &\Psi(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)) \\ &\qquad = f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x). \end{aligned}$$

Consequently, any $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code C can be generated as a $\mathbb{Z}_{p^s}[x]$-submodule of $R_{\alpha,\beta,\gamma,\eta}$ with the following form

$$\begin{aligned} C = \big\langle & \big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0; 0\big)\,, \\ & \big(\ell_1(x); f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x); 0; 0\big)\,, \\ & \big(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2q_2(x) + \cdots + p^{s-1}q_{s-1}(x); 0\big)\,, \\ & \big(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)\big)\,\big\rangle. \end{aligned}$$

□

Now, the following theorem can be deduced from Theorem 3.1 in a manner entirely similar to the way in which the corresponding result was deduced from [12, Theorem 3.3].

Theorem 3.2. *Assume that C is a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code. Then we may assume that*

$$\begin{aligned} C = \big\langle & \big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0; 0\big)\,, \\ & \big(\ell_1(x); f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x); 0; 0\big)\,, \\ & \big(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2q_2(x) + \cdots + p^{s-1}q_{s-1}(x); 0\big)\,, \\ & \big(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)\big)\,\big\rangle, \end{aligned}$$

where

$$\begin{aligned} &\deg(\ell_1(x)) < \deg\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x)\big), \\ &\deg(\ell_2(x)) < \deg\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x)\big) \text{ and} \\ &\deg(\ell_4(x)) < \deg\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x)\big). \end{aligned}$$

Lemma 3.3. *If*

$$\begin{aligned} C = \big\langle & \big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0; 0\big)\,, \\ & \big(\ell_1(x); f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x); 0; 0\big)\,, \\ & \big(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2q_2(x) + \cdots + p^{s-1}q_{s-1}(x); 0\big)\,, \\ & \big(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)\big)\,\big\rangle \end{aligned}$$

is a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code, then we may assume that

(i) $\big(f_2(x) + pb_1(x) + \cdots + p^{r-1}b_{r-1}(x)\big) \,\big|\, \frac{x^{\gamma}-1}{q_{s-1}}\ell_3(x) \mod p^r$,

(ii) $\big(f_3(x) + pq_1(x) + \cdots + p^{s-1}q_{s-1}(x)\big) \,\big|\, \frac{x^{\eta}-1}{g_{s-1}}\ell_6(x) \mod p^s$, *and*

(iii) $\big(f_1(x) + pa_1(x) + \cdots + p^{r-1}a_{r-1}(x)\big) \,\big|\, k(x)\ell_1(x) - \frac{x^{\gamma}-1}{q_{s-1}}\ell_2(x) \bmod p^r$,

where $k(x)\big(f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x)\big) = \frac{x^{\gamma}-1}{q_{s-1}}\ell_3(x)$.

Proof. (i) Consider the multiplication

$$\frac{x^\gamma-1}{q_{s-1}(x)}*\big(\ell_2(x);\ell_3(x);f_3(x)+pq_1(x)+p^2q_2(x)+\cdots+p^{s-1}q_{s-1}(x);0\big)$$
$$=\Big(\frac{x^\gamma-1}{q_{s-1}(x)}\ell_2(x);\frac{x^\gamma-1}{q_{s-1}(x)}\ell_3(x);0;0\Big).$$

Since $\Psi\Big(\frac{x^\gamma-1}{q_{s-1}(x)}\ell_2(x);\frac{x^\gamma-1}{q_{s-1}(x)}\ell_3(x);0;0\Big)=0$, we have

$$\Psi\Big(\frac{x^\gamma-1}{q_{s-1}(x)}\ell_2(x);\frac{x^\gamma-1}{q_{s-1}(x)}\ell_3(x);0;0\Big)\in \mathrm{Ker}(\Psi)\subseteq C,$$

and so

$$\big(f_2(x)+pb_1(x)+p^2b_2(x)+\cdots+p^{r-1}b_{r-1}(x)\big)\Big|\ \frac{x^\gamma-1}{q_{s-1}}\ell_3(x)\mod p^r.$$

(ii) We consider the following equality.

$$\frac{x^\eta-1}{g_{s-1}(x)}*\big(\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{s-1}g_{s-1}(x)\big)$$
$$=\Big(\frac{x^\eta-1}{g_{s-1}(x)}\ell_4(x);\frac{x^\eta-1}{g_{s-1}(x)}\ell_5(x);\frac{x^\eta-1}{g_{s-1}(x)}\ell_6(x);0\Big)$$

Since $\Psi\Big(\frac{x^\eta-1}{g_{s-1}(x)}\ell_4(x);\frac{x^\eta-1}{g_{s-1}(x)}\ell_5(x);\frac{x^\eta-1}{g_{s-1}(x)}\ell_6(x);0\Big)=0$, we have

$$\Psi\Big(\frac{x^\eta-1}{g_{s-1}(x)}\ell_4(x);\frac{x^\eta-1}{g_{s-1}(x)}\ell_5(x);\frac{x^\eta-1}{g_{s-1}(x)}\ell_6(x);0\Big)\in \mathrm{Ker}(\Psi)\subseteq C,$$

and so

$$\big(f_3(x)+pq_1(x)+p^2q_2(x)+\cdots+p^{s-1}q_{s-1}(x)\big)\Big|\ \frac{x^\eta-1}{g_{s-1}}\ell_6(x)\mod p^s.$$

(iii) Since

$$\big(f_2(x)+pb_1(x)+p^2b_2(x)+\cdots+p^{r-1}b_{r-1}(x)\big)\Big|\frac{x^\gamma-1}{b_{r-1}(x)}\ell_3(x)\mod p^r,$$

we can write

$$\big(f_2(x)+pb_1(x)+p^2b_2(x)+\cdots+p^{r-1}b_{r-1}(x)\big)k(x)=\frac{x^\gamma-1}{b_{r-1}(x)}\ell_3(x)$$

for some $k(x)\in\mathbb{Z}_{p^r}[x]$. Now, consider the multiplication

$$k(x)*\big(\ell_1(x);f_2(x)+pb_1(x)+p^2b_2(x)+\cdots+p^{r-1}b_{r-1}(x);0;0\big)$$
$$=\big(k(x)\ell_1(x);k(x)(f_2(x)+pb_1(x)+p^2b_2(x)+\cdots+p^{r-1}b_{r-1}(x));0;0\big)\in C.$$

On the other hand, $\left(\frac{x^\gamma-1}{q_{s-1}(x)}\ell_2(x); \frac{x^\gamma-1}{q_{s-1}(x)}\ell_3(x); 0; 0\right) \in C$. Hence,

$$\begin{aligned}
&\big(k(x)\ell_1(x); k(x)(f_2(x)+pb_1(x)+p^2b_2(x)+\cdots+p^{r-1}b_{r-1}(x)); 0; 0\big)\\
&-\Big(\frac{x^\gamma-1}{q_{s-1}(x)}\ell_2(x); \frac{x^\gamma-1}{q_{s-1}(x)}\ell_3(x); 0; 0\Big)\\
&=\Big(k(x)\ell_1(x)-\frac{x^\gamma-1}{q_{s-1}(x)}\ell_2(x); 0; 0; 0\Big) \in \mathrm{Ker}(\Psi) \subseteq C.
\end{aligned}$$

Thus $\big(f_1(x)+pa_1(x)+p^2a_2(x)+\cdots+p^{r-1}a_{r-1}(x)\big)\big|k(x)\ell_1(x)-\frac{x^\gamma-1}{q_{s-1}(x)}\ell_2(x) \mod p^r$. □

Theorem 3.4. *Let the notation be as in Lemma 3.3. Then*

$$\begin{aligned}
C_\alpha &= \big\langle \gcd\big(f_1(x)+pa_1(x)+p^2a_2(x)+\cdots+p^{r-1}a_{r-1}(x), \ell_1(x), \ell_2(x), \ell_4(x)\big)\big\rangle,\\
C_\beta &= \big\langle \gcd\big(f_2(x)+pb_1(x)+p^2b_2(x)+\cdots+p^{r-1}b_{r-1}(x), \ell_3(x), \ell_5(x)\big)\big\rangle,\\
C_\gamma &= \big\langle \gcd\big(f_3(x)+pq_1(x)+p^2q_2(x)+\cdots+p^{s-1}q_{s-1}(x), \ell_6(x)\big)\big\rangle,\\
C_\eta &= \big\langle f_4(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{s-1}g_{s-1}(x)\big\rangle, \text{ and also}\\
(C_\alpha)^\perp &= \Big\langle \frac{x^\alpha-1}{\gcd\big((f_1(x)+pa_1(x)+\cdots+p^{r-1}a_{r-1}(x))^*, \ell_1^*(x), \ell_2^*(x), \ell_4^*(x)\big)}\Big\rangle,\\
(C_\beta)^\perp &= \Big\langle \frac{x^\beta-1}{\gcd\big((f_2(x)+pb_1(x)+\cdots+p^{r-1}b_{r-1}(x))^*, \ell_3^*(x), \ell_5^*(x)\big)}\Big\rangle,\\
(C_\gamma)^\perp &= \Big\langle \frac{x^\gamma-1}{\gcd\big((f_3(x)+pq_1(x)+\cdots+p^{s-1}q_{s-1}(x))^*, \ell_6^*(x)\big)}\Big\rangle \text{ and}\\
(C_\eta)^\perp &= \big\langle (f_4(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{s-1}g_{s-1}(x))^*\big\rangle,
\end{aligned}$$

where

$$\begin{aligned}
&a_{r-1}(x)|a_{r-2}(x)|\dots|a_1(x)|f_1(x)|(x^\alpha-1) \bmod p^r,\\
&b_{r-1}(x)|b_{r-2}(x)|\dots|b_1(x)|f_2(x)|(x^\beta-1) \bmod p^r,\\
&q_{s-1}(x)|q_{s-2}(x)|\dots|q_1(x)|f_3(x)|(x^\gamma-1) \bmod p^s, \text{ and}\\
&g_{s-1}(x)|g_{s-2}(x)|\dots|g_1(x)|f_4(x)|(x^\eta-1) \bmod p^s.
\end{aligned}$$

Here, for a polynomial $h(x)$, $h^*(x)$ denotes the reciprocal polynomial of $h(x)$.

Proof. Assume that $c_1(x) \in C_\alpha$. Then there exist $c_2(x) \in \mathbb{Z}_{p^r}[x]/\langle x^\beta-1\rangle$, $c_3(x) \in \mathbb{Z}_{p^s}[x]/\langle x^\gamma-1\rangle$ and $c_4(x) \in \mathbb{Z}_{p^s}[x]/\langle x^\eta-1\rangle$ such that $\big(c_1(x); c_2(x); c_3(x); c_4(x)\big) \in C$. It follows that there exist $d_1(x), d_2(x), d_3(x), d_4(x) \in \mathbb{Z}_{p^s}[x]$ such that

$$\begin{aligned}
c(x) = {}& d_1(x) * \big(f_1(x)+pa_1(x)+p^2a_2(x)+\cdots+p^{r-1}a_{r-1}(x); 0; 0; 0\big)\\
&+ d_2(x) * \big(\ell_1(x); f_2(x)+pb_1(x)+p^2b_2(x)+\cdots+p^{r-1}b_{r-1}(x); 0; 0\big)\\
&+ d_3(x) * \big(\ell_2(x); \ell_3(x); f_3(x)+pq_1(x)+p^2q_2(x)+\cdots+p^{s-1}q_{s-1}(x); 0\big)\\
&+ d_4(x) * \big(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{s-1}g_{s-1}(x)\big).
\end{aligned}$$

Clearly, the equality

$$\begin{aligned}c_1(x) = d_1(x) * \big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x)\big)\\ + d_2(x) * \ell_1(x) + d_3(x) * \ell_2(x) + d_4(x) * \ell_4(x),\end{aligned}$$

implies that $\gcd\big(f_1(x)+pa_1(x)+p^2a_2(x)+\cdots+p^{r-1}a_{r-1}(x), \ell_1(x), \ell_2(x), \ell_4(x)\big)\big| c_1(x)$. Thus $c_1(x) \in \big\langle \gcd\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x), \ell_1(x), \ell_2(x), \ell_4(x)\big)\big\rangle$, and so we can conclude that

$$C_\alpha \subseteq \big\langle \gcd\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x), \ell_1(x), \ell_2(x), \ell_4(x)\big)\big\rangle.$$

On the other hand, there exist $\mu_1(x), \mu_2(x), \mu_3(x), \mu_4(x) \in \mathbb{Z}_{p^r}[x]$ such that

$$\begin{aligned}&\gcd\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x), \ell_1(x), \ell_2(x), \ell_4(x)\big) =\\ &\mu_1(x)\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x)\big) + \mu_2(x)\ell_1(x) + \mu_3(x)\ell_2(x)\\ &+ \mu_4(x)\ell_4(x).\end{aligned}$$

Then

$$\begin{aligned}&\big(\gcd\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x), \ell_1(x), \ell_2(x), \ell_4(x)\big);\\ &\mu_2(x)(f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x)) + \mu_3(x)\ell_3(x) + \mu_4(x)\ell_5(x);\\ &\mu_3(x)(f_3(x) + pq_1(x) + p^2q_2(x) + \cdots + p^{s-1}q_{s-1}(x)) + \mu_4(x)\ell_6(x);\\ &\mu_4(x)(f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x))\\ &= \mu_1(x)\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0; 0\big)\\ &+ \mu_2(x)\big(\ell_1(x); f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x); 0; 0\big)\\ &+ \mu_3(x)\big(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2q_2(x) + \cdots + p^{s-1}q_{s-1}(x); 0\big)\\ &+ \mu_4(x)\big(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)\big) \subseteq C.\end{aligned}$$

This implies that

$$\big\langle \gcd\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); \ell_1(x); \ell_2(x); \ell_4(x)\big)\big\rangle \subseteq C_\alpha,$$

and so $C_\alpha = \big\langle \gcd\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); \ell_1(x); \ell_2(x); \ell_4(x)\big)\big\rangle$. Similarly, one can prove the next three claims about generating sets of C_β, C_γ and C_η. Since C_α is a cyclic code of order p^r, by using the theory of dual cyclic codes, we have that

$$(C_\alpha)^\perp = \Big\langle \frac{x^\alpha - 1}{\gcd\big((f_1(x) + pa_1(x) + \cdots + p^{r-1}a_{r-1}(x))^*; \ell_1^*(x); \ell_2^*(x); \ell_4^*(x)\big)}\Big\rangle.$$

Similarly, one can prove the next three claims about generating sets of $(C_\beta)^\perp$, $(C_\gamma)^\perp$ and $(C_\eta)^\perp$. Also, in view of [13, Proposition 3.4], it is easy to see that $\gcd(a(x), b(x))^* = \gcd(a^*(x), b^*(x))$. This completes the proof. □

In the next theorem, we summarize our results about $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic codes.

Theorem 3.5. *Let C be a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code. Then we can classify C as follows:*

(i) $C = \big\langle \big(\big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0; 0\big)\big) \big\rangle$, *where* $a_{r-1}(x)|a_{r-2}(x)|\ldots|a_1(x)|f_1(x)|(x^\alpha - 1) \bmod p^r$.

(ii) $C = \big\langle \big(\ell_1(x); f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x); 0; 0\big) \big\rangle$, *where* $b_{r-1}(x)|b_{r-2}(x)|\ldots|b_1(x)|f_2(x)|(x^\beta - 1) \bmod p^r$, *and* $\ell_1(x)$ *is a polynomial in* $\mathbb{Z}_{p^r}[x]$.

(iii) $C = \big\langle \big(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2q_2(x) + \cdots + p^{s-1}q_{s-1}(x); 0\big) \big\rangle$, *where* $q_{s-1}(x)|q_{s-2}(x)|\ldots|q_1(x)|f_3(x)|(x^\gamma - 1) \bmod p^s$, *and* $\ell_2(x)$, $\ell_3(x)$ *are polynomials in* $\mathbb{Z}_{p^r}[x]$.

(iv) $C = \big\langle \big(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)\big) \big\rangle$, *where* $g_{s-1}(x)|g_{s-2}(x)|\ldots|g_1(x)|f_4(x)|(x^\eta - 1) \bmod p^s$, *and* $\ell_4(x)$, $\ell_5(x)$ *are polynomials in* $\mathbb{Z}_{p^r}[x]$ *and* $\ell_6(x)$ *is a polynomial in* $\mathbb{Z}_{p^s}[x]$.

(v)

$$\begin{aligned} C = \big\langle\, & \big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0; 0\big)\,, \\ & \big(\ell_1(x); f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x); 0; 0\big)\,, \\ & \big(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2q_2(x) + \cdots + p^{s-1}q_{s-1}(x); 0\big)\,, \\ & \big(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)\big) \big\rangle, \end{aligned}$$

where the generator polynomials are defined in Theorem 3.1.

4 Minimal generating sets

Let C be a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code. Then C is also a $\mathbb{Z}_{p^s}[x]$-module. In this section, we will determine the minimum generating sets of C in $R_{\alpha,\beta,\gamma,\eta}$ as a $\mathbb{Z}_{p^s}[x]$-module. These sets will be used to determine the size of C.

Let $C = \big\langle g_0, pg_1, \ldots, p^{m-1}g_{m-1}\big\rangle$ be a cyclic code of length n and $g = g_0 + pg_1 + \cdots + p^{m-1}g_{m-1}$. Since g_0 is a factor of $x^n - 1$ and, for $i = 1, \ldots, m-1$, the polynomial g_i is a factor of g_{i-1}, we may define the polynomials $\hat{g}_0 := \frac{x^n-1}{g_0}$ and $\hat{g}_i := \frac{g_{i-1}}{g_i}$, for $i = 1, \ldots, m-1$. Define $G := \prod_{i=0}^{m-1} \hat{g}_i$. It is clear that $G_g := (\prod_{i=0}^{m-1} \hat{g}_i)g = 0$ over $\mathbb{Z}_{p^m}[x]/\langle x^n - 1\rangle$.

Theorem 4.1. *Let*

$$\begin{aligned} C = \big\langle\, & \big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0; 0\big)\,, \\ & \big(\ell_1(x); f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x); 0; 0\big)\,, \\ & \big(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2q_2(x) + \cdots + p^{s-1}q_{s-1}(x); 0\big)\,, \\ & \big(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)\big) \big\rangle \end{aligned}$$

be a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic code with the generator polynomials defined in Theorem 3.1. Define the sets

$$A_i := \left\{ x^m (\prod_{y=0}^{i-1} \hat{a}_y) * \big(f_1(x) + pa_1(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0; 0\big) \right\}_{m=0}^{\deg(\hat{a}_i)-1}$$
$$\text{for} \quad 0 \le i \le r-1, \text{and}$$
$$B_j := \left\{ x^m (\prod_{y=0}^{j-1} \hat{b}_y) * \big(\ell_1(x); f_2(x) + pb_1(x) + \cdots + p^{r-1}b_{r-1}(x); 0; 0\big) \right\}_{m=0}^{\deg(\hat{b}_j)-1}$$
$$\text{for} \quad 0 \le j \le r-1, \text{and}$$
$$Q_k := \left\{ x^m (\prod_{y=0}^{k-1} \hat{q}_y) * \big(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + \cdots + p^{s-1}q_{s-1}(x); 0\big) \right\}_{m=0}^{\deg(\hat{q}_k)-1}$$
$$\text{for} \quad 0 \le k \le s-1, \text{and}$$
$$G_u := \left\{ x^m (\prod_{y=0}^{u-1} \hat{g}_y) * \big(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + \cdots + p^{s-1}g_{s-1}(x)\big) \right\}_{m=0}^{\deg(\hat{g}_u)-1}$$
$$\text{for} \quad 0 \le u \le s-1.$$

Then

$$S := \left(\bigcup_{i=0}^{r-1} A_i\right) \bigcup \left(\bigcup_{j=0}^{r-1} B_j\right) \bigcup \left(\bigcup_{k=0}^{s-1} Q_k\right) \bigcup \left(\bigcup_{u=0}^{s-1} G_u\right)$$

forms a minimal generating set for C as a $\mathbb{Z}_{p^t}[x]$-module. Moreover,

$$|C| = p^{\sum_{i=0}^{r-1}(r-i)\deg(\hat{a}_i) + \sum_{j=0}^{r-1}(r-j)\deg(\hat{b}_j) + \sum_{k=0}^{s-1}(s-k)\deg(\hat{q}_k) + \sum_{u=0}^{s-1}(s-u)\deg(\hat{g}_u)}.$$

Proof. In view of [3, Theorem 2.5], it is clear that the elements in S are $\mathbb{Z}_{p^s}$-linearly independent, because $\left(\bigcup_{i=0}^{r-1} A_i\right)_\alpha$, $\left(\bigcup_{j=0}^{r-1} B_j\right)_\beta$, $\left(\bigcup_{k=0}^{s-1} Q_k\right)_\gamma$ and $\left(\bigcup_{u=0}^{s-1} G_u\right)_\eta$ are minimal generating sets for the codes C_α, C_β, C_γ and C_η, respectively.

Assume that $c(x)$ is an arbitrary codeword in C. Then

$$\begin{aligned} c(x) &= \mu(x) * \big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0; 0\big) \\ &\quad + d(x) * \big(\ell_1(x); f_2(x) + pb_1(x) + p^2b_2(x) + \cdots + p^{r-1}b_{r-1}(x); 0; 0\big) \\ &\quad + e(x) * \big(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2q_2(x) + \cdots + p^{s-1}q_{s-1}(x); 0\big) \\ &\quad + w(x) * \big(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + p^2g_2(x) + \cdots + p^{s-1}g_{s-1}(x)\big), \end{aligned}$$

for some polynomials $\mu(x), d(x), e(x)$ and $w(x)$ over $\mathbb{Z}_{p^s}$. The same argument as in [3, Theorem 2.5] shows that

$$\mu(x) * \big(f_1(x) + pa_1(x) + p^2a_2(x) + \cdots + p^{r-1}a_{r-1}(x); 0; 0; 0\big) \in \left\langle \bigcup_{i=0}^{r-1} A_i \right\rangle_{\mathbb{Z}_{p^r}}. \quad (1)$$

So we have to prove the following properties:

$$d(x) * \left(\ell_1(x); f_2(x) + pb_1(x) + p^2 b_2(x) + \cdots + p^{r-1} b_{r-1}(x); 0; 0\right) \in \langle S \rangle_{\mathbb{Z}_{p^r}}, \quad (2)$$

$$e(x) * \left(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2 q_2(x) + \cdots + p^{s-1} q_{s-1}(x); 0\right) \in \langle S \rangle_{\mathbb{Z}_{p^s}} \quad (3)$$

and

$$w(x) * \left(\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pg_1(x) + \cdots + p^{s-1} g_{s-1}(x)\right) \in \langle S \rangle_{\mathbb{Z}_{p^s}}. \quad (4)$$

If $\deg(d(x)) < \deg\left(\frac{x^\beta - 1}{f_2(x)}\right)$, then

$$d(x) * \left(\ell_1(x); f_2(x) + pb_1(x) + p^2 b_2(x) + \cdots + p^{r-1} b_{r-1}(x); 0; 0\right) \in \left\langle B_0 \right\rangle_{\mathbb{Z}_{p^r}}.$$

Otherwise, we set $d(x) := \frac{x^\beta - 1}{f_2(x)} d_0(x) + r_0(x)$ with $\deg(r_0(x)) < \deg\left(\frac{x^\beta - 1}{f_2(x)}\right)$. Then,

$$\begin{aligned} d(x) * \left(\ell_1(x); f_2(x) + pb_1(x) + p^2 b_2(x) + \cdots + p^{r-1} b_{r-1}(x); 0; 0\right) = \\ \frac{x^\beta - 1}{f_2(x)} d_0(x) * \left(\ell_1(x); f_2(x) + pb_1(x) + p^2 b_2(x) + \cdots + p^{r-1} b_{r-1}(x); 0; 0\right) \\ + r_0(x) * \left(\ell_1(x); f_2(x) + pb_1(x) + p^2 b_2(x) + \cdots + p^{r-1} b_{r-1}(x); 0; 0\right), \end{aligned}$$

and so

$$r_0(x) * \left(\ell_1(x); f_2(x) + pb_1(x) + p^2 b_2(x) + \cdots + p^{r-1} b_{r-1}(x); 0; 0\right) \in \left\langle B_0 \right\rangle_{\mathbb{Z}_{p^r}}.$$

Thus

$$\begin{aligned} \frac{x^\beta - 1}{f_2(x)} d_0(x) * \left(\ell_1(x); f_2(x) + pb_1(x) + p^2 b_2(x) + \cdots + p^{r-1} b_{r-1}(x); 0; 0\right) \\ = \left(\frac{x^\beta - 1}{f_2(x)} d_0(x) \ell_1(x); 0; 0; 0\right). \end{aligned}$$

By (1), we have that $\left(\frac{x^\beta - 1}{f_2(x)} d_0(x) \ell_1(x); 0; 0; 0\right) \in \left\langle \bigcup_{i=0}^{r-1} A_i \right\rangle_{\mathbb{Z}_{p^r}}$. Then

$$d(x) * \left(\ell_1(x); f_2(x) + pb_1(x) + p^2 b_2(x) + \cdots + p^{r-1} b_{r-1}(x); 0; 0\right)$$

is an element in $\left\langle \bigcup_{j=0}^{r-1} B_j \right\rangle_{\mathbb{Z}_{p^r}} \bigcup \left\langle \bigcup_{i=0}^{r-1} A_i \right\rangle_{\mathbb{Z}_{p^r}}$. Thus (2) is established. In the following, we prove (3). It must be proved that

$$e(x) * \left(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2 q_2(x) + \cdots + p^{s-1} q_{s-1}(x); 0\right) \in \langle S \rangle_{\mathbb{Z}_{p^s}}.$$

To achieve this aim, if $\deg(e(x)) < \deg\left(\frac{x^\gamma - 1}{f_3(x)}\right)$, then

$$e(x) * \left(\ell_2(x); \ell_3(x); f_3(x) + pq_1(x) + p^2 q_2(x) + \cdots + p^{s-1} q_{s-1}(x); 0\right) \in \langle Q_0 \rangle_{\mathbb{Z}_{p^s}}.$$

Otherwise, we put $e(x) := e_0(x)\left(\frac{x^\gamma-1}{f_3(x)}\right) + r_0(x)$, where $\deg(r_0(x)) < \deg\left(\frac{x^\gamma-1}{f_3(x)}\right)$. Then

$$\begin{aligned} e(x) * \big(\ell_2(x);\ell_3(x);f_3(x)+pq_1(x)+p^2q_2(x)+\cdots+p^{s-1}q_{s-1}(x);0\big) &= \\ e_0(x)(\frac{x^\gamma-1}{f_3(x)}) * \big(\ell_2(x);\ell_3(x);f_3(x)+pq_1(x)+p^2q_2(x)+\cdots+p^{s-1}q_{s-1}(x);0\big) \\ +r_0(x) * \big(\ell_2(x);\ell_3(x);f_3(x)+pq_1(x)+p^2q_2(x)+\cdots+p^{s-1}q_{s-1}(x);0\big)\,, \end{aligned}$$

and hence

$$r_0(x) * \big(\ell_2(x);\ell_3(x);f_3(x)+pq_1(x)+p^2q_2(x)+\cdots+p^{s-1}q_{s-1}(x);0\big) \in \langle Q_0\rangle_{\mathbb{Z}_{p^s}}\,.$$

Thus

$$\begin{aligned} e_0(x)(\frac{x^\gamma-1}{f_3(x)}) * \big(\ell_2(x);\ell_3(x);f_3(x)+pq_1(x)+p^2q_2(x)+\cdots+p^{s-1}q_{s-1}(x);0\big) \\ = \left(\ell_2(x)e_0(x)(\frac{x^\gamma-1}{f_3(x)});\ell_3(x)e_0(x)(\frac{x^\gamma-1}{f_3(x)});0;0\right). \end{aligned}$$

By (2), we have $\left(\ell_2(x)e_0(x)(\frac{x^\gamma-1}{f_3(x)});\ell_3(x)e_0(x)(\frac{x^\gamma-1}{f_3(x)});0;0\right) \in \langle S\rangle_{\mathbb{Z}_{p^r}}$. Then

$$e(x) * \big(\ell_1(x);\ell_2(x);f_3(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{t-1}g_{t-1}(x)\big)$$

is an element of $\left\langle\bigcup_{k=0}^{s-1}Q_k\right\rangle_{\mathbb{Z}_{p^s}} \bigcup \left\langle\bigcup_{j=0}^{r-1}B_j\right\rangle_{\mathbb{Z}_{p^r}} \bigcup \left\langle\bigcup_{i=0}^{r-1}A_i\right\rangle_{\mathbb{Z}_{p^r}}$. Now, we just have to show that

$$w(x) * \big(\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{s-1}g_{s-1}(x)\big) \in \langle S\rangle_{\mathbb{Z}_{p^s}}\,.$$

If $\deg(w(x)) < \deg\left(\frac{x^\eta-1}{f_4(x)}\right)$, then

$$w(x) * \big(\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{s-1}g_{s-1}(x)\big) \in \langle G_0\rangle_{\mathbb{Z}_{p^s}}\,,$$

and so $c(x) \in \langle S\rangle_{\mathbb{Z}_{p^s}}$. Otherwise, we put $w(x) := w_0(x)\left(\frac{x^\eta-1}{f_4(x)}\right) + r_0(x)$, where $\deg(r_0(x)) < \deg\left(\frac{x^\eta-1}{f_4(x)}\right)$. Then

$$\begin{aligned} & w(x) * \big(\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{s-1}g_{s-1}(x)\big) \\ & = w_0(x)(\frac{x^\eta-1}{f_4(x)}) * \big(\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{s-1}g_{s-1}(x)\big) \\ & + r_0(x) * \big(\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{s-1}g_{s-1}(x)\big)\,, \end{aligned}$$

and hence

$$r_0(x) * \big(\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{s-1}g_{s-1}(x)\big) \in \langle G_0\rangle_{\mathbb{Z}_{p^s}}\,.$$

Thus

$$\begin{aligned} & w_0(x)(\frac{x^\eta-1}{f_4(x)}) * \big(\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{s-1}g_{s-1}(x)\big) \\ &= \left(\ell_4(x)w_0(x)(\frac{x^\eta-1}{f_4(x)});\ell_5(x)w_0(x)(\frac{x^\eta-1}{f_4(x)});\ell_6(x)w_0(x)(\frac{x^\eta-1}{f_4(x)});0\right). \end{aligned}$$

By (3), we have $\left(\ell_4(x)w_0(x)(\frac{x^\eta-1}{f_4(x)});\ell_5(x)w_0(x)(\frac{x^\eta-1}{f_4(x)});\ell_6(x)w_0(x)(\frac{x^\eta-1}{f_4(x)});0\right) \in \langle S\rangle_{\mathbb{Z}_{p^s}}$. Then

$$w(x) * \big(\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pg_1(x)+p^2g_2(x)+\cdots+p^{s-1}g_{s-1}(x)\big)$$

is an element of $\langle S\rangle_{\mathbb{Z}_{p^r}} \cup \langle S\rangle_{\mathbb{Z}_{p^s}} \cup \langle S\rangle_{\mathbb{Z}_{p^s}}$. Thus $c(x) \in \langle S\rangle_{\mathbb{Z}_{p^s}}$, and so S is a minimal generating set for C. □

Example 4.2. *(The notation is as in Theorem 4.1.) Let C be a $\mathbb{Z}_2\mathbb{Z}_8$-double cyclic code with $p=2, r=1, s=3, \alpha=9, \beta=10, \gamma=2$, and $\eta=7$, generated by*

$$\begin{aligned} &\{(f_1(x);0;0;0),(\ell_1(x);f_2(x);0;0),(\ell_2(x);\ell_3(x);f_3(x)+2q_1(x)+4q_2(x);0), \\ &(\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+2g_1(x)+4g_2(x))\}, \end{aligned}$$

where

$$\begin{aligned} f_1(x) &= x^6+x^3+1, \\ f_2(x) &= x^4+x^3+x^2+x+1, \\ f_3(x) &= 0, \\ f_4(x) &= x^4+2x^3+7x^2+5x+1, \\ \ell_i(x) &= 0, \text{ for } i=1,\ldots,6, \\ q_1(x) &= q_2(x)=0, g_1(x)=x^3+3x^2+2x+7, g_2(x)=1. \end{aligned}$$

Hence

$$\begin{aligned} \deg(\hat{a_0}) &= \deg(\frac{x^9-1}{x^6+x^3+1}) = \deg(x^3-1) = 3, \\ \deg(\hat{b_0}) &= \deg(\frac{x^{10}-1}{x^4+x^3+x^2+x+1}) = \deg(x^6-x^5+x-1) = 6, \\ \deg(\hat{q_0}) &= \deg(\hat{q_1}) = \deg(\hat{q_2}) = 0, \\ \deg(\hat{g_0}) &= \deg(\frac{x^7-1}{x^4+2x^3+7x^2+5x+1}) = \deg(x^3+6x^2+5x+7) = 3, \\ \deg(\hat{g_1}) &= \deg(\frac{x^4+2x^3+7x^2+5x+1}{x^3+3x^2+2x+7}) = \deg(x+7) = 1, \\ \deg(\hat{g_2}) &= \deg(\frac{x^3+3x^2+2x+7}{1}) = \deg(x^3+3x^2+2x+7) = 3. \end{aligned}$$

We put

$$S := \left(\bigcup_{i=0}^{r-1} A_i\right) \bigcup \left(\bigcup_{j=0}^{r-1} B_j\right) \bigcup \left(\bigcup_{k=0}^{s-1} Q_k\right) \bigcup \left(\bigcup_{u=0}^{s-1} G_u\right).$$

Then $A_0 \cup B_0 \cup G_0 \cup G_1 \cup G_2$ *forms a minimum generating set for* C, *where*

$$\begin{aligned}
A_0 &:= \{x^i(x^3-1) * (x^6+x^3+1;0;0;0)\}_{i=0}^{2},\\
B_0 &:= \{x^i(x^6-x^5+x-1) * (0;x^4+x^3+x^2+x+1;0;0)\}_{i=0}^{5},\\
G_0 &:= \{x^i(x^3+6x^2+5x+7) * (0;0;;0;x^4+4x^3+5x^2+x+3)\}_{i=0}^{2},\\
G_1 &:= \{(x^3+6x^2+5x+7) * (0;0;;0;x^4+4x^3+5x^2+x+3)\},\\
G_2 &:= \{x^i(x+7)(x^3+6x^2+5x+7) * (0;0;;0;x^4+4x^3+5x^2+x+3)\}_{i=0}^{2}.
\end{aligned}$$

Moreover, $|C| = p^{\sum_{i=0}^{r-1}(r-i)\deg(\hat{a}_i)+\sum_{j=0}^{r-1}(r-j)\deg(\hat{b}_j)+\sum_{k=0}^{s-1}(s-k)\deg(\hat{q}_k)+\sum_{u=0}^{s-1}(s-u)\deg(\hat{g}_u)}$. *Hence* C *has* 2^{23} *codewords and* $n = 75$.

5 Generator polynomials

In this section, we focus on the case where $r = 1$ and $s = 2$, and study submodules of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-double cyclic code. Also, we obtain generator polynomials of this family of codes.

Definition 5.1. *In [17], the classical Gray map* $\phi : \mathbb{Z}_{p^2} \to \mathbb{Z}_p^p$ *is defined as*

$$\phi(\theta) = \theta_0(1,1,\ldots,1) + \theta_1(0,1,\ldots,p-1),$$

where $\theta = \theta_0 p + \theta_1 \in \mathbb{Z}_{p^2}, 0 \le \theta_0, \theta_1 < p$ *and* $\phi(\theta)$ *is a vector of length* p.

For any $x = (x_1, x_2, \ldots, x_r) \in \mathbb{Z}_p^r$, $y = (y_1, y_2, \ldots, y_s) \in \mathbb{Z}_p^s$, $z = (z_1, z_2, \ldots, z_t) \in \mathbb{Z}_{p^2}^t$, $w = (w_1, w_2, \ldots, w_u) \in \mathbb{Z}_{p^2}^u$ *and* $n = r + s + p(t+u)$, *the map*

$$\Phi : \mathbb{Z}_p^{r+s} \times \mathbb{Z}_{p^2}^{t+u} \to \mathbb{Z}_p^n$$

is defined by

$$\Phi(x;y;z;w) = \big(x_1,\ldots,x_r,y_1,\ldots,y_s,\phi(z_1),\ldots,\phi(z_t),\phi(w_1),\ldots,\phi(w_u)\big).$$

Clearly the map Φ is one-to-one but not surjective, and so, in general, it is not bijective. Moreover, according to Proposition 2.6, a subset C of $R_{r,s,t,u}$ is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-double cyclic code if and only if C is a $\mathbb{Z}_{p^2}[x]$-submodule of $R_{r,s,t,u}$. Also, by a method similar that we used in Theorems 3.1 and 3.2 together with Lemma 3.3, we have the following result.

Theorem 5.2. *Let* C *be a* $\mathbb{Z}_p\mathbb{Z}_{p^2}$*-double cyclic code. Then*

$$\begin{aligned}
C = \big\langle\, &(f_1(x);0;0;0)\,, (\ell_1(x);f_2(x);0;0)\,, (\ell_2(x);\ell_3(x);f_3(x)+pa(x);0)\,,\\
&(\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pq(x))\,\big\rangle,
\end{aligned}$$

where $f_1(x), f_2(x), \ell_i(x) \in \mathbb{Z}_p[x]$, *for* $i = 1,\ldots,5$, *and* $f_3(x), f_4(x), \ell_6(x), a(x), q(x) \in \mathbb{Z}_{p^2}[x]$ *which satisfy the conditions*

$$\begin{aligned}
&f_1(x)|(x^r-1) \bmod p,\\
&f_2(x)|(x^s-1) \bmod p,\\
&a(x)|f_3(x)|(x^t-1) \bmod p^2, \text{ and}\\
&q(x)|f_4(x)|(x^u-1) \bmod p^2.
\end{aligned}$$

Also, we may assume that

$$\deg(\ell_1(x)) < \deg\big(f_1(x)\big), \deg(\ell_2(x)) < \deg\big(f_1(x)\big), \deg(\ell_4(x)) < \deg\big(f_1(x)\big),$$

and that

$$\big(f_3(x) + pa(x)\big) \Big| \frac{x^u - 1}{q(x)} \ell_6(x) \mod p^2.$$

Theorem 5.3. *Let*

$$C = \big\langle\, (f_1(x); 0; 0; 0)\,, (\ell_1(x); f_2(x); 0; 0)\,, (\ell_2(x); \ell_3(x); f_3(x) + pa(x); 0)\,, \\ (\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pq(x))\,\big\rangle,$$

be a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-double cyclic code. Then

$$\begin{aligned}
&C_r = \big\langle \gcd\big(f_1(x), \ell_1(x), \ell_2(x), \ell_4(x)\big)\big\rangle,\\
&C_s = \big\langle \gcd\big(f_2(x), f_3(x), \ell_5(x)\big)\big\rangle,\\
&C_t = \big\langle \gcd\big(f_3(x) + pa(x), \ell_6(x)\big)\big\rangle,\\
&C_u = \big\langle f_4(x) + pq(x)\big\rangle,\\
&(C_r)^\perp = \Big\langle \frac{x^r - 1}{\gcd\big(f_1^*(x), \ell_1^*(x), \ell_2^*(x), \ell_4^*(x)\big)}\Big\rangle,\\
&(C_s)^\perp = \Big\langle \frac{x^s - 1}{\gcd\big(f_2^*(x), \ell_3^*(x), \ell_5^*(x)\big)}\Big\rangle,\\
&(C_t)^\perp = \Big\langle \frac{x^t - 1}{\gcd\big((f_3(x) + pa(x))^*, \ell_6^*(x)\big)}\Big\rangle \text{ and } (C_u)^\perp = \Big\langle \frac{x^u - 1}{\gcd\big(h^*(x)g^*(x) + ph^*(x)\big)}\Big\rangle,
\end{aligned}$$

where $q(x)|f_4(x)|(x^u - 1) \mod p^2$, $f_4(x) = f(x)g(x)$, $q(x) = g(x), x^u - 1 = f(x)g(x)h(x)$ and $g(x)|f(x)g(x)|f(x)g(x)h(x)$. Here, for a polynomial $h(x)$, $h^*(x)$ denotes its reciprocal polynomial.

Proof. The proof similar to that we used to prove Theorem 3.4. □

In the next theorem, we summarize our results about $\mathbb{Z}_p\mathbb{Z}_{p^2}$-double cyclic codes.

Theorem 5.4. *Let C be a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-double cyclic code. Then we can classify C as follows:*

(i) $C = \big\langle\, (f_1(x); 0; 0; 0)\,\big\rangle$, *where* $f_1(x)|(x^r - 1) \mod p$.

(ii) $C = \big\langle\, (\ell_1(x); f_2(x); 0; 0)\,\big\rangle$, *where* $f_2(x)|(x^s - 1) \mod p$.

(iii) $C = \big\langle\, (\ell_2(x); \ell_3(x); f_3(x) + pa(x); 0)\,\big\rangle$, *where* $a(x)|f_3(x)|(x^t - 1) \mod p^2$ *and* $\ell_2(x)$, $\ell_3(x)$ *are polynomials in* $\mathbb{Z}_p[x]$.

(iv) $C = \big\langle\, (\ell_4(x); \ell_5(x); \ell_6(x); f_4(x) + pq(x))\,\big\rangle$, *where* $q(x)|f_4(x)|(x^u - 1) \mod p^2$ *and* $\ell_4(x)$, $\ell_5(x)$ *are polynomials in* $\mathbb{Z}_p[x]$ *and* $\ell_6(x)$ *is a polynomial in* $\mathbb{Z}_{p^2}[x]$.

(v) $C = \big\langle\, (f_1(x); 0; 0; 0)\,, (\ell_1(x); f_2(x); 0; 0)\,\big\rangle$, *where* $f_1(x)|(x^r - 1) \mod p$, $f_2(x)|(x^s - 1) \mod p$ *and* $\ell_1(x)$ *is a polynomial in* $\mathbb{Z}_p[x]$.

(vi) $C = \left\langle (f_1(x);0;0;0), (\ell_2(x);\ell_3(x);f_3(x)+pa(x);0) \right\rangle$, *where* $f_1(x)|(x^r-1) \mod p$, $a(x)|f_3(x)|(x^t-1) \mod p^2$ *and* $\ell_2(x),\ell_3(x)$ *are polynomials in* $\mathbb{Z}_p[x]$.

(vii) $C = \left\langle (f_1(x);0;0;0), (\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pq(x)) \right\rangle$, *where* $f_1(x)|(x^r-1) \mod p$, $q(x)|f_4(x)|(x^u-1) \mod p^2$ *and* $\ell_4(x),\ell_5(x)$ *are polynomials in* $\mathbb{Z}_p[x]$ *and* $\ell_6(x)$ *is a polynomial in* $\mathbb{Z}_{p^2}[x]$.

(viii) $C = \left\langle (\ell_1(x);f_2(x);0;0), (\ell_2(x);\ell_3(x);f_3(x)+pa(x);0) \right\rangle$, *where* $f_2(x)|(x^s-1) \mod p$, $a(x)|f_3(x)|(x^t-1) \mod p^2$ *and* $\ell_1(x),\ell_2(x),\ell_3(x)$ *are polynomials in* $\mathbb{Z}_p[x]$.

(ix) $C = \left\langle (\ell_1(x);f_2(x);0;0), (\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pq(x)) \right\rangle$, *where* $f_2(x)|(x^s-1) \mod p$, $q(x)|f_4(x)|(x^u-1) \mod p^2$ *and* $\ell_1(x),\ell_4(x),\ell_5(x)$ *are polynomials in* $\mathbb{Z}_p[x]$ *and* $\ell_6(x)$ *is a polynomial in* $\mathbb{Z}_{p^2}[x]$.

(x) $C = \left\langle (\ell_2(x);\ell_3(x);f_3(x)+pa(x);0), (\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pq(x)) \right\rangle$, *where* $a(x)|f_3(x)|(x^t-1) \mod p^2$, $q(x)|f_4(x)|(x^u-1) \mod p^2$ *and* $\ell_1(x),\ell_3(x),\ell_4(x),\ell_5(x)$ *are polynomials in* $\mathbb{Z}_p[x]$ *and also* $\ell_6(x)$ *is a polynomial in* $\mathbb{Z}_{p^2}[x]$.

(xi) $C = \left\langle (f_1(x);0;0;0), (\ell_1(x);f_2(x);0;0), (\ell_2(x);\ell_3(x);f_3(x)+pa(x);0) \right\rangle$, *where* $f_1(x)|(x^r-1) \mathrm{mod}\ p$, $f_2(x)|(x^s-1) \mathrm{mod}\ p$, $a(x)|f_3(x)|(x^t-1) \mod p^2$ *and* $\ell_1(x),\ell_2(x),\ell_3(x)$ *are polynomials in* $\mathbb{Z}_p[x]$

(xii) $C = \left\langle (f_1(x);0;0;0), (\ell_1(x);f_2(x);0;0), (\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pq(x)) \right\rangle$, *where* $f_1(x)|(x^r-1) \mathrm{mod}\ p$, $f_2(x)|(x^s-1) \mathrm{mod}\ p$, $q(x)|f_4(x)|(x^u-1) \mod p^2$ *and* $\ell_1(x),\ell_4(x),\ell_5(x)$ *are polynomials in* $\mathbb{Z}_p[x]$ *and also* $\ell_6(x)$ *is a polynomial in* $\mathbb{Z}_{p^2}[x]$.

(xiii)

$$C = \left\langle (f_1(x);0;0;0), (\ell_2(x);\ell_3(x);f_3(x)+pa(x);0), \right.$$
$$\left. (\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pq(x)) \right\rangle,$$

where $f_1(x)|(x^r-1)\mathrm{mod}\ p$, $a(x)|f_3(x)|(x^t-1)\mathrm{mod}\ p^2$, $q(x)|f_4(x)|(x^u-1) \mod p^2$ *and* $\ell_2(x),\ell_4(x),\ell_5(x)$ *are polynomials in* $\mathbb{Z}_p[x]$ *and also* $\ell_6(x)$ *is a polynomial in* $\mathbb{Z}_{p^2}[x]$.

(xiv)

$$C = \left\langle (\ell_1(x);f_2(x);0;0), (\ell_2(x);\ell_3(x);f_3(x)+pa(x);0), \right.$$
$$\left. (\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pq(x)) \right\rangle,$$

where $f_2(x)|(x^s-1)\mathrm{mod}\ p$, $a(x)|f_3(x)|(x^t-1)\mathrm{mod}\ p^2$, $q(x)|f_4(x)|(x^u-1) \mod p^2$ *and* $\ell_1(x),\ell_2(x),\ell_4(x),\ell_5(x)$ *are polynomials in* $\mathbb{Z}_p[x]$ *and also* $\ell_6(x)$ *is a polynomial in* $\mathbb{Z}_{p^2}[x]$.

(xv)

$$C = \left\langle (f_1(x);0;0;0), (\ell_1(x);f_2(x);0;0), (\ell_2(x);\ell_3(x);f_3(x)+pa(x);0), \right.$$
$$\left. (\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pq(x)) \right\rangle,$$

where the generator polynomials were defined in Theorem 5.2.

6 Minimal generating sets

Theorem 6.1. *Let*

$$C = \left\langle \, (f_1(x);0;0;0)\, , (\ell_1(x);f_2(x);0;0)\, , (\ell_2(x);\ell_3(x);f_3(x)+pa(x);0)\, , \right.$$
$$\left. (\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pq(x)) \, \right\rangle,$$

be a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-double cyclic code with the generator polynomials defined in Theorem 3.1 with

$$\begin{aligned}
&\deg(f_1(x)) = r_1,\ \deg(f_2(x)) = s_1,\\
&\deg(f_3(x)) = t_1,\ \deg(a(x)) = t_2,\\
&\deg(f_4(x)) = u_1,\ \deg(q(x)) = u_2,\\
&h_1(x) = \frac{x^t-1}{f_3(x)} \text{ and } h_2(x) = \frac{x^u-1}{f_4(x)}.
\end{aligned}$$

Define the sets

$$\begin{aligned}
S_1 &= \bigcup_{i=0}^{r-r_1-1} \left\{x^i * (f_1(x);0;0;0)\right\},\\
S_2 &= \bigcup_{i=0}^{s-s_1-1} \left\{x^i * (\ell_1(x);f_2(x);0;0)\right\},\\
S_3 &= \bigcup_{i=0}^{t-t_1-1} \left\{x^i * (\ell_2(x);\ell_3(x);f_3(x)+pa(x);0)\right\},\\
S_4 &= \bigcup_{i=0}^{t_1-t_2-1} \left\{x^i * (h_1(x)\ell_2(x);h_1(x)\ell_3(x);pa(x)h_1(x);0)\right\},\\
S_5 &= \bigcup_{i=0}^{u-u_1-1} \left\{x^i * (\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+pq(x))\right\} \text{ and}\\
S_6 &= \bigcup_{i=0}^{u_1-u_2-1} \left\{x^i * (h_2(x)\ell_4(x);h_2(x)\ell_5(x);h_2(x)\ell_6(x);pq(x)h_2(x))\right\},
\end{aligned}$$

Then, $S = \cup_{i=1}^{6} S_i$ forms a minimal generating set for C as a $\mathbb{Z}_{p^2}[x]$-module. Moreover,

$$|C| = p^{r-r_1}p^{s-s_1}p^{t-t_1}p^{u-u_1}p^{2^{t_1-t_2}}p^{2^{u_1-u_2}}.$$

Proof. The proof similar to that we used to prove Theorem 4.1. ∎

Example 6.2. *Let C be $\mathbb{Z}_3\mathbb{Z}_9$-double cyclic code with $p=3, r=2, s=7, t=3,$ and $u=11$, generated by*

$$\{(f_1(x);0;0;0), (\ell_1(x);f_2(x);0;0), (\ell_2(x);\ell_3(x);f_3(x)+3a(x);0),$$
$$(\ell_4(x);\ell_5(x);\ell_6(x);f_4(x)+3q(x))\},$$

where

$$\begin{aligned}
f_1(x) &= x+2,\\
f_2(x) &= x^6+x^5+x^4+x^3+x^2+x+1,\\
f_3(x) &= x^2-5x+7,\\
f_4(x) &= x-1,\\
\ell_1(x) &= \ell_2(x) = 1,\\
\ell_i(x) &= 0 \text{ for } i=3,\dots,6, \text{ and}\\
a(x) &= q(x) = 1.
\end{aligned}$$

Now, since $f_3(x)h_1(x) = x^3-1$ *and* $f_4(x)h_2(x) = x^{11}-1$, *we have that*

$$\begin{aligned}
h_1(x) &= x+5\\
h_2(x) &= x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1
\end{aligned}$$

and also $r_1 = 1, s_1 = 6, t_1 = 2, t_2 = 0, u_1 = 1, u_2 = 0$.

Thus

$$\begin{aligned}
S_1 &= \{(x+2;0;0;0)\},\\
S_2 &= \left\{(1;x^6+x^5+x^4+x^3+x^2+x+1;0;0)\right\},\\
S_3 &= \left\{(1;0;x^2-5x+1;0)\right\},\\
S_4 &= \bigcup_{i=0}^{1}\left\{x^i*(x+5;0;3x+15;0)\right\},\\
S_5 &= \bigcup_{i=0}^{9}\left\{x^i*(0;0;0;x+2)\right\},\\
S_6 &= \left\{\left(0;0;0;3(x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1)\right)\right\}.
\end{aligned}$$

Hence

$$S := \bigcup_{i=0}^{6} S_i$$

forms a minimum generating set for C. *Moreover,* C *has* 3^{19} *codewords and* $n = 51$.

7 Conclusions

In this paper, we studied the algebraic structure of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic codes, where $r \leq s$. These codes can be viewed as $\mathbb{Z}_{p^s}[x]$-submodules of the module

$$\mathbb{Z}_{p^r}/<x^\alpha-1>\times\mathbb{Z}_{p^r}/<x^\beta-1>\times\mathbb{Z}_{p^s}/<x^\gamma-1>\times\mathbb{Z}_{p^s}/<x^\eta-1>.$$

Moreover, we study the generator polynomials of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-double cyclic codes. We also discuss the p-ary images (Gray images) of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-double cyclic codes.

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