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A family of q-analogues for a congruence by Sun

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Abstract

Recently, Guo conjectured two q-analogues for a congruence discovered by Sun, which were subsequently verified by Liu and Qi. In this paper, we provide a family of q-analogues for the same Sun's congruence by utilizing Carlitz's identity and a q-analogue of Morley's congruence.

Key Words: Combinatorial congruence, q-congruence, cyclotomic polynomial, q-supercongruence, Carlitz's formula.
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1 Introduction

In 2010, Sun and Tauraso [13, Corollary 1.1] proved the following congruence for central binomial coefficients: for any odd prime p and positive integer r,

$$\sum_{k=0}^{p^r-1}\frac{1}{2^k}\binom{2k}{k}\equiv(-1)^{\frac{p^r-1}{2}}\pmod{p}. \tag{1.1}$$

Sun [12, Corollary 1.1] further demonstrated in the same year that (1.1) is also true modulo p^2, i.e.,

$$\sum_{k=0}^{p^r-1}\frac{1}{2^k}\binom{2k}{k}\equiv(-1)^{\frac{p^r-1}{2}}\pmod{p^2}. \tag{1.2}$$

Later, Guo and Zeng [5, Corollary 4.2] and Guo [3, Theorem 1.1] respectively gave the following q-analogues of (1.1) and (1.2): for any positive odd integer n,

$$\sum_{k=0}^{n-1}\frac{(q;q^2)_k}{(q;q)_k}q^k\equiv(-1)^{\frac{n-1}{2}}q^{\frac{n^2-1}{4}}\pmod{\Phi_n(q)}, \tag{1.3}$$

$$\sum_{k=0}^{n-1}\frac{(q;q^2)_k}{(q;q)_k}q^k\equiv(-1)^{\frac{n-1}{2}}q^{\frac{n^2-1}{4}}\pmod{\Phi_n(q)^2}. \tag{1.4}$$

Here the *q-shifted factorial* is defined as $(a;q)_n:=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\in\mathbb{Z}^+$ and $(a;q)_0:=1$, and the *n-th cyclotomic polynomial* $\Phi_n(q)$ is given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k)$$

with ζ being an n-th primitive root of unity.

Furthermore, Wang and Yu established a generalization of (1.3) in [16, Theorem 1.1] and they also verified an extension of (1.4) in [17, Theorem 1] as the following q-congruences: for any integer d and positive odd integer n with $n > 2|d| - 1$,

$$\sum_{k=0}^{n-1} \frac{(q^{2d+1};q^2)_k}{(q;q)_k} q^k \equiv (-1)^{\frac{n-1}{2}+d} q^{\frac{n^2-(2d+1)^2}{4}} \pmod{\Phi_n(q)},$$

$$\sum_{k=0}^{n-1} \frac{(q^{2d+1};q^2)_k}{(q;q)_k} q^k \equiv (-1)^{\frac{n-1}{2}+d} q^{\frac{n^2-(2d+1)^2}{4}} + \operatorname{Sgn}(d) \sum_{t=1}^{|d|} \frac{(-1)^{d-t}(1+q^{2t-1})[n]}{q^{2\binom{d+1}{2}-2\binom{t}{2}}[2t-1]} \pmod{\Phi_n(q)^2},$$

where $\operatorname{Sgn}(d)$ is the Sign function and $[n] := [n]_q = (1-q^n)/(1-q)$ is the *q-integer*. It is worth noting that the $d = \pm 1$ cases of the first q-congruence were presented by Gu and Guo [2], while the $d = \pm 1$ cases of the second q-congruence were provided by Wang and Ni [15].

Recently, Guo [4, Theorem 1] established two new q-analogues of (1.1): for any positive odd integer n,

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{2k} \equiv (-1)^{\frac{n-1}{2}} \pmod{\Phi_n(q)},$$

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k} q^{2k+1} \equiv (-1)^{\frac{n-1}{2}} \pmod{\Phi_n(q)}.$$

He further conjectured that, for any positive odd integer n, modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{2k} \equiv \begin{cases} q^{\binom{n}{2}}, & \text{if } n \equiv 1 \pmod{4}, \\ -q^{\binom{n+1}{2}}, & \text{if } n \equiv 3 \pmod{4}, \end{cases} \tag{1.5}$$

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k} q^{2k+1} \equiv \begin{cases} q^{\binom{n+1}{2}}, & \text{if } n \equiv 1 \pmod{4}, \\ -q^{\binom{n}{2}}, & \text{if } n \equiv 3 \pmod{4}, \end{cases} \tag{1.6}$$

which were just confirmed by Liu and Qi [10, Theorem 1.2]. The investigation of q-congruence has garnered significant attention from experts. For more results, the interested readers are referred to [1, 2, 6, 7, 14, 15].

Motivated by the aforementioned work, we extend (1.5) and (1.6) by establishing the following q-supercongruence.

Theorem 1. *Let n be a positive odd integer and d an integer. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-q^{2d};q^2)_k}{(q^2;q^2)_k}q^{2k+d} \tag{1.7}$$

$$\equiv\begin{cases}(q^n-1)\left(-d+2\mathrm{Sgn}(d)\displaystyle\sum_{i=1}^{|d|}\frac{q^{i+t}}{1+q^{2(i+t)}}\right)+q^{dn+\binom{n}{2}}, & \text{if } n\equiv 1 \pmod 4,\\[2ex] (q^n-1)\left(d+2\mathrm{Sgn}(d)\displaystyle\sum_{i=1}^{|d|}\frac{q^{i+t}}{1+q^{2(i+t)}}\right)-q^{dn+\binom{n+1}{2}}, & \text{if } n\equiv 3 \pmod 4,\end{cases}$$

where $t=-(\mathrm{Sgn}(d)+1)/2$.

Putting $n=p^r$ with odd prime p and positive integer r, then taking $q\to 1$ in Theorem 1, we get (1.2). Obviously, the q-supercongruences (1.5) and (1.6) can be respectively obtained by making $d=0$ and $d=1$ in Theorem 1.

2 Proof of Theorem 1

In this section, a q-analogue of Morley's congruence introduced by Liu, Pan and Zhang [9] and Carlitz's identity (2.5) will be employed to prove Theorem 1. Before giving the proof of Theorem 1, we first present the following result as a preliminary lemma.

Lemma 1. *Let d be an integer and n a positive odd integer. Then, modulo $\Phi_n(q)$,*

$$\sum_{\substack{0\le k\le (n-1)\\ k\ne\frac{n-1}{2}}}(-1)^{k+1}\frac{q^{d(2k+1)}}{1-q^{2k+1}}\equiv 2\mathrm{Sgn}(d)\sum_{i=1}^{|d|}\frac{q^{i+t}}{1+q^{2(i+t)}}+\frac{(-1)^{\frac{n-3}{2}}(2d-1)-1}{2},$$

where $t=-(\mathrm{Sgn}(d)+1)/2$.

Proof. We will prove this lemma by considering three different cases of d.

- $d=0$. Obviously, the $d=0$ case of Lemma 1 happens to be the following result:

$$\sum_{\substack{0\le k\le (n-1)\\ k\ne\frac{n-1}{2}}}(-1)^{k}\frac{1}{1-q^{2k+1}}\equiv\frac{1+(-1)^{\frac{n-3}{2}}}{2}\pmod{\Phi_n(q)}, \tag{2.1}$$

which can be found in Liu and Qi [10, (2.13)].

- $d > 0$. We have

$$\begin{aligned}
&\sum_{\substack{0\le k\le (n-1)\\ k\neq \frac{n-1}{2}}} (-1)^k \frac{1}{1-q^{2k+1}} + \sum_{\substack{0\le k\le (n-1)\\ k\neq \frac{n-1}{2}}} (-1)^{k+1} \frac{q^{d(2k+1)}}{1-q^{2k+1}} \\
=&\sum_{\substack{0\le k\le (n-1)\\ k\neq \frac{n-1}{2}}} (-1)^k \left(\sum_{i=1}^{d} q^{(2k+1)(i-1)}\right) \\
=&\sum_{i=1}^{d}\left(\frac{q^{i-1}\left(1+q^{2(i-1)n}\right)}{1+q^{2(i-1)}} - (-1)^{\frac{n-1}{2}} q^{(i-1)n}\right) \\
\equiv& 2\sum_{i=1}^{d}\left(\frac{q^{i-1}}{1+q^{2(i-1)}}\right) + (-1)^{\frac{n-3}{2}} d \pmod{\Phi_n(q)},
\end{aligned} \tag{2.2}$$

where we have utilized the fact that $q^n \equiv 1 \pmod{\Phi_n(q)}$ in the last step. Combining the two results (2.1) and (2.2), we immediately establish the $d > 0$ case of Lemma 1.

- $d < 0$. Similar to the proof for $d > 0$, we can arrive at the subsequent result:

$$\begin{aligned}
&\sum_{\substack{0\le k\le (n-1)\\ k\neq \frac{n-1}{2}}} (-1)^k \frac{1}{1-q^{2k+1}} + \sum_{\substack{0\le k\le (n-1)\\ k\neq \frac{n-1}{2}}} (-1)^{k+1} \frac{q^{d(2k+1)}}{1-q^{2k+1}} \\
=&-\sum_{i=1}^{-d}\left(\frac{q^{i+d-1}\left(1+q^{2n(i+d-1)}\right)}{1+q^{2(i+d-1)}} - (-1)^{\frac{n-1}{2}} q^{n(i+d-1)}\right) \\
\equiv&-2\sum_{i=1}^{-d}\left(\frac{q^{i}}{1+q^{2i}}\right) + (-1)^{\frac{n-3}{2}} d \pmod{\Phi_n(q)}.
\end{aligned} \tag{2.3}$$

We finish the proof for $d < 0$ of Lemma 1 by subtracting (2.1) from (2.3). This completes the proof of Lemma 1. ◻

Proof of Theorem 1. Dividing by q^d on both sides of (1.7) and then replacing q by q^{-1}, we find that the q-congruence (1.7) is equivalent to the following result: modulo $\Phi_n(q)^2$,

$$\begin{aligned}
&\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-q^{2d};q^2)_k}{(q^2;q^2)_k} q^{-2dk-k^2} \\
\equiv&\begin{cases}
q^d(1-q^n)\left(-d + 2\mathrm{Sgn}(d)\displaystyle\sum_{i=1}^{|d|}\frac{q^{i+t}}{1+q^{2(i+t)}}\right) + q^{-d(n-1)-\binom{n}{2}}, & \text{if } n \equiv 1 \pmod{4}, \\
q^d(1-q^n)\left(d + 2\mathrm{Sgn}(d)\displaystyle\sum_{i=1}^{|d|}\frac{q^{i+t}}{1+q^{2(i+t)}}\right) - q^{-d(n-1)-\binom{n+1}{2}}, & \text{if } n \equiv 3 \pmod{4},
\end{cases}
\end{aligned} \tag{2.4}$$

with $t = -(\mathrm{Sgn}(d)+1)/2$. Therefore, we only need to show that the q-supercongruence (2.4) holds for $d > 0$ and $d < 0$, for the $d = 0$ case of (1.7) is just (1.5).

Recall Carlitz's identity [1]:

$$\sum_{k=0}^{n} \frac{(a;q)_k (b;q)_k}{(q;q)_k} (-ab)^{n-k} q^{\frac{(n-k)(n+k-1)}{2}} = \sum_{k=0}^{n} \frac{(a;q)_{n+1} (-b)^k q^{\binom{k}{2}}}{(q;q)_k (q;q)_{n-k} (1-aq^{n-k})}. \tag{2.5}$$

By making the substitutions $q \to q^2, a = q, b = -q^{2d}$ and $n \to n-1$ in (2.5), we are led to the following identity:

$$\begin{aligned} &\sum_{k=0}^{n-1} \frac{(q;q^2)_k (-q^{2d};q^2)_k}{(q^2;q^2)_k} q^{(n-1)^2-k^2+2d(n-1-k)} \\ =&\sum_{k=0}^{n-1} \frac{(q;q^2)_n q^{k^2-k+2dk}}{(q^2;q^2)_k (q^2;q^2)_{n-1-k} (1-q^{2n-2k-1})} \\ =&\sum_{k=0}^{n-1} \frac{(q;q^2)_n q^{k^2-k+2dk}}{(q^2;q^2)_{n-1} (1-q^{2n-2k-1})} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2}. \end{aligned} \tag{2.6}$$

For convenience, we will denote the k-th term on the right-hand side of (2.6) briefly by $a_{n,k}$, namely,

$$a_{n,k} = \frac{(q;q^2)_n q^{k^2-k+2dk}}{(q^2;q^2)_{n-1} (1-q^{2n-2k-1})} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2}, \tag{2.7}$$

where the q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

As n is an odd positive integer, $(q;q^2)_n$ contains the factor $\Phi_n(q)$ and $(q^2;q^2)_{n-1}$ is coprime with $\Phi_n(q)$. Also $1-q^{2n-2k-1}$ is coprime with $\Phi_n(q)$ except for $k = \frac{n-1}{2}$. Employing the fact that $q^n \equiv 1 \pmod{\Phi_n(q)}$, we get, for $0 \le k \le n-1$,

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} = \prod_{i=1}^{k} \frac{1-q^{2n-2i}}{1-q^{2i}} \equiv \prod_{i=1}^{k} \frac{1-q^{-2i}}{1-q^{2i}} = (-1)^k q^{-k^2-k} \pmod{\Phi_n(q)}.$$

Now, we can derive the following q-congruence from (2.7)

$$a_{n,k\left(k \neq \frac{n-1}{2}\right)} \equiv (-1)^k \frac{(1-q)(q;q^2)_{n-1}}{(q^2;q^2)_{n-1}(1-q^{2k+1})} q^{2dk} \pmod{\Phi_n(q)^2}. \tag{2.8}$$

Meanwhile, the following two relations [15, (2.4) and (2.7)]

$$\frac{(q;q^2)_{n-1}}{(q;q)_{n-1}} \equiv -q[n] \pmod{\Phi_n(q)^2}, \qquad (-q;q)_{n-1} \equiv 1 \pmod{\Phi_n(q)},$$

imply that

$$\frac{(q;q^2)_{n-1}}{(q^2;q^2)_{n-1}}=\frac{(q;q^2)_{n-1}}{(q;q)_{n-1}(-q;q)_{n-1}}\equiv -q[n]\pmod{\Phi_n(q)^2}.$$

Then, the q-congruence (2.8) can be simplified as

$$a_{n,k\left(k\neq\frac{n-1}{2}\right)}\equiv(-1)^{k+1}\frac{1-q^n}{1-q^{2k+1}}q^{2dk+1}\pmod{\Phi_n(q)^2}. \tag{2.9}$$

In [11], Pan showed that

$$(-1)^{\frac{n-1}{2}}q^{\frac{n^2-1}{4}}{\begin{bmatrix}n-1\\ \frac{n-1}{2}\end{bmatrix}}_{q^2}\equiv(-q;q)_{n-1}^2-\frac{n^2-1}{24}(1-q)^2[n]_q^2\pmod{\Phi_n(q)^3},$$

which means

$${\begin{bmatrix}n-1\\ \frac{n-1}{2}\end{bmatrix}}_{q^2}\equiv(-1)^{\frac{n-1}{2}}q^{\frac{1-n^2}{4}}(-q;q)_{n-1}^2\pmod{\Phi_n(q)^2}.$$

Utilizing the above q-congruence, we can infer the following conclusion,

$$\begin{aligned}a_{n,\frac{n-1}{2}}&=\frac{(q;q^2)_nq^{\frac{n^2-4n+3}{4}+d(n-1)}}{(q^2;q^2)_{n-1}(1-q^n)}{\begin{bmatrix}n-1\\ \frac{n-1}{2}\end{bmatrix}}_{q^2}\\&\equiv(-1)^{\frac{n-1}{2}}\frac{(q;q^2)_n(-q;q)_{n-1}^2}{(1-q^n)(q^2;q^2)_{n-1}}q^{(d-1)(n-1)}\pmod{\Phi_n(q)^2}.\end{aligned}\tag{2.10}$$

Additionally, note that the following q-congruences [8, (4.4)]:

$$\begin{bmatrix}2n\\ n\end{bmatrix}\equiv 2-n(1-q^n)\pmod{\Phi_n(q)^2}\quad\text{and}\quad\frac{1}{1+q^n}\equiv\frac{1}{2}+\frac{1}{4}(1-q^n)\pmod{\Phi_n(q)^2},$$

it is not difficult to find that

$$\frac{(q;q^2)_n(-q;q)_{n-1}^2}{(1-q^n)(q^2;q^2)_{n-1}}=\frac{\begin{bmatrix}2n\\ n\end{bmatrix}}{1+q^n}\equiv 1-\frac{(n-1)(1-q^n)}{2}\pmod{\Phi_n(q)^2}.$$

Therefore, we can simplify (2.10) as follows:

$$a_{n,\frac{n-1}{2}}\equiv(-1)^{\frac{n-1}{2}}q^{(d-1)(n-1)}\left(1-\frac{(n-1)(1-q^n)}{2}\right)\pmod{\Phi_n(q)^2}. \tag{2.11}$$

In conclusion, from (2.6), (2.7), (2.9) and (2.11) we deduce that, for an integer d and a positive odd integer n, modulo $\Phi_n(q)^2$,

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-q^{2d};q^2)_k}{(q^2;q^2)_k}q^{(n-1)^2-k^2+2d(n-1-k)}\\&\equiv\sum_{\substack{0\le k\le(n-1)\\ k\neq\frac{n-1}{2}}}(-1)^{k+1}\frac{1-q^n}{1-q^{2k+1}}q^{2dk+1}+(-1)^{\frac{n-1}{2}}q^{(d-1)(n-1)}\left(1-\frac{(n-1)(1-q^n)}{2}\right).\end{aligned}\tag{2.12}$$

Now, we shall distinguish four cases of d and n to verify the truth of (2.4).

• $d>0$, $n\equiv 1 \pmod 4$. With the assistance of Lemma 1 and (2.12), we can make the following deduction, modulo $\Phi_n(q)^2$,

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-q^{2d};q^2)_k}{(q^2;q^2)_k}q^{(n-1)^2-k^2+2d(n-1-k)}\\&\equiv q^{1-d}(1-q^n)\left(-d+2\sum_{i=1}^{d}\frac{q^{i-1}}{1+q^{2(i-1)}}\right)+q^{(d-1)(n-1)}\left(1-\frac{(n-1)(1-q^n)}{2}\right).\end{aligned}\tag{2.13}$$

It is straightforward to show that

$$1-m(1-q^n)\equiv 1-(1-q^n)(1+q^n+q^{2n}+\cdots+q^{(m-1)n})\equiv q^{mn}\pmod{\Phi_n(q)^2}.\tag{2.14}$$

Consequently, (2.13) yields

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-q^{2d};q^2)_k}{(q^2;q^2)_k}q^{(n-1)^2-k^2+2d(n-1-k)}\\&\equiv q^{1-d}(1-q^n)\left(-d+2\sum_{i=1}^{d}\frac{q^{i-1}}{1+q^{2(i-1)}}\right)+q^{(n-1)(n+d-1)-\binom{n}{2}}\pmod{\Phi_n(q)^2}.\end{aligned}\tag{2.15}$$

Now we can observe the equivalence between (2.15) and (2.4) by $q^n\equiv 1 \pmod{\Phi_n(q)}$ with $d>0$ and $n\equiv 1 \pmod 4$.

• $d>0$, $n\equiv 3 \pmod 4$. Lemma 1 and (2.12) allow us to establish the subsequent result, modulo $\Phi_n(q)^2$,

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-q^{2d};q^2)_k}{(q^2;q^2)_k}q^{(n-1)^2-k^2+2d(n-1-k)}\\&\equiv q^{1-d}(1-q^n)\left(d+2\sum_{i=1}^{d}\frac{q^{i-1}}{1+q^{2(i-1)}}\right)-q^{(d-1)(n-1)}\left(1-\frac{(n-3)(1-q^n)}{2}\right)\\&\equiv q^{1-d}(1-q^n)\left(d+2\sum_{i=1}^{d}\frac{q^{i-1}}{1+q^{2(i-1)}}\right)-q^{(n+d-1)(n-1)-\binom{n+1}{2}}.\end{aligned}$$

Here we also have utilized (2.14). Then the proof for the $d>0$ and $n\equiv 3 \pmod 4$ case of (2.4) is completed.

• $d<0$, $n\equiv 1 \pmod 4$. We can similarly infer the following outcome by utilizing Lemma 1, (2.12) and (2.14), modulo $\Phi_n(q)^2$,

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-q^{2d};q^2)_k}{(q^2;q^2)_k}q^{(n-1)^2-k^2+2d(n-1-k)}\\&\equiv q^{1-d}(1-q^n)\left(-d-2\sum_{i=1}^{-d}\frac{q^{i}}{1+q^{2i}}\right)+q^{(d-1)(n-1)}\left(1-\frac{(n-1)(1-q^n)}{2}\right)\\&\equiv q^{1-d}(1-q^n)\left(-d-2\sum_{i=1}^{-d}\frac{q^{i}}{1+q^{2i}}\right)+q^{(n-1)(n+d-1)-\binom{n}{2}}.\end{aligned}$$

We complete the proof for the $d<0$ and $n\equiv 1 \pmod 4$ case of (2.4).

- $d<0$, $n\equiv 3 \pmod 4$. Likewise, applying Lemma 1, (2.12) and (2.14), we get the following result, modulo $\Phi_n(q)^2$,

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-q^{2d};q^2)_k}{(q^2;q^2)_k}q^{(n-1)^2-k^2+2d(n-1-k)}\\&\equiv q^{1-d}(1-q^n)\left(d-2\sum_{i=1}^{-d}\frac{q^i}{1+q^{2i}}\right)-q^{(d-1)(n-1)}\left(1-\frac{(n-3)(1-q^n)}{2}\right)\\&\equiv q^{1-d}(1-q^n)\left(d-2\sum_{i=1}^{-d}\frac{q^i}{1+q^{2i}}\right)-q^{(n+d-1)(n-1)-\binom{n+1}{2}}.\end{aligned}$$

We get the $d<0$ and $n\equiv 3 \pmod 4$ case of (2.4). □

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