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Attached primes of top local cohomology modules
by

Gholamreza Pirmohammadi

Abstract

Let $\mathfrak{a}$ be an ideal of a Noetherian complete local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module with $\operatorname{cd}(\mathfrak{a}, M) = c \geq 1$. Assume that the R-module $H^c_{\mathfrak{a}}(M)$ is Artinian. In this paper, it is shown that

$$\operatorname{Att}_R H^c_{\mathfrak{a}}(M) \subseteq \operatorname{mAss}_R M \cup \{\mathfrak{p} \in \operatorname{Supp}_R M : \operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p} = \operatorname{Ann}_R H^c_{\mathfrak{a}}(R/\mathfrak{p})\}.$$

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1 Introduction

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and $\mathfrak{a}$ be an ideal of R. In this paper, we will denote $\operatorname{Supp} R/\mathfrak{a} = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. In addition, the symbol $\mathbb{N}$ (respectively $\mathbb{N}_0$) will denote the set of positive (respectively non-negative) integers. The ith local cohomology module of an R-module M with support in $V(\mathfrak{a})$ is defined as:

$$H^i_{\mathfrak{a}}(M) = \varinjlim_{n \in \mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

For further information on the concept of local cohomology, the reader may consult [8] and [13].

Recall that an R-module M is called $\mathfrak{a}$-*cofinite* if $\operatorname{Supp} M \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i \geq 0$. The concept of $\mathfrak{a}$-cofinite modules were introduced by Hartshorne [14] and have been studied by several authors; see e.g. [4, 5, 6, 7, 9, 10, 11, 12, 16, 17, 18, 19, 20, 22, 23, 25].

Recall that for an R-module M its cohomological dimension with respect to $\mathfrak{a}$, denoted by $\operatorname{cd}(\mathfrak{a}, M)$ is defined as the supremum of all integers i such that $H^i_{\mathfrak{a}}(M) \neq 0$.

A prime ideal $\mathfrak{p}$ is said to be *attached* to M if $\mathfrak{p} = \operatorname{Ann}_R M/N$ for some submodule N of M, equivalently $\mathfrak{p} = \operatorname{Ann}_R M/\mathfrak{p}M$. If $(R, \mathfrak{m})$ is a Noetherian complete local ring, the attached primes of an Artinian module are precisely the associated primes of its Matlis dual. In the sequel, we denote the set of all attached primes of M by $\operatorname{Att}_R M$. Note that all attached primes of a module M contain the annihilator of M.

Let M be a finitely generated R-module with $\operatorname{cd}(\mathfrak{a}, M) = c \geq 1$. It is well know that $\operatorname{Att}_R H^c_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \operatorname{Supp}_R M : \operatorname{Ann}_R H^c_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p}\}$; see [1, Theorem 2.2]. Now, consider the special case that the R-module $H^c_{\mathfrak{a}}(M)$ is Artinian. To the best of our knowledge, there is not any other special explicit description of the set $\operatorname{Att}_R H^c_{\mathfrak{a}}(M)$ in the literature for this special case. So, at least for the case that the R-module $H^c_{\mathfrak{a}}(M)$ is Artinian, the elements of the set $\operatorname{Att}_R H^c_{\mathfrak{a}}(M)$ deserves a deeper investigation. Of course, if the R-module $H^c_{\mathfrak{a}}(M)$ is Artinian and $\mathfrak{a}$-cofinite then by [2, Theorems 2.1, 2.10], we can find a useful alternative description of the set $\operatorname{Att}_R H^c_{\mathfrak{a}}(M)$. But in general, under our assumptions, $H^c_{\mathfrak{a}}(M)$ does not need to be $\mathfrak{a}$-cofinite; see [4, Example 2.5(i)]. Although, the top local cohomology module $H^c_{\mathfrak{a}}(M)$ is Artinian and $\mathfrak{a}$-cofinite, whenever $c = \dim M$; see [23, Proposition 5.1].

Pursuing this point of view further, in this paper we establish the following inclusion

$$\operatorname{Att}_R H^c_{\mathfrak{a}}(M) \subseteq \operatorname{mAss}_R M \cup \{\mathfrak{p} \in \operatorname{Supp}_R M : \operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p} = \operatorname{Ann}_R H^c_{\mathfrak{a}}(R/\mathfrak{p})\},$$

whenever $(R, \mathfrak{m})$ is a Noetherian complete local ring and the R-module $H^c_{\mathfrak{a}}(M)$ is Artinian.

Throughout this paper, for each R-module L, the set of minimal elements of $\operatorname{Ass}_R L$ with respect to inclusion is denoted by $\operatorname{mAss}_R L$. Also, for each R-module M, we denote the *Krull's dimension* of M by $\dim_R M$. For each R-module M of finite length, we denote the length of M by $\ell_R(M)$. We refer the reader to [8, 21] for basic results, notations, and terminology not given in this paper.

2 The results

We start this section with some useful well known lemmas.

Lemma 2.1. *(See [1, Theorem 2.2]) Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and M be a finitely generated R-module with $\operatorname{cd}(\mathfrak{a}, M) = c \geq 0$. Then*

$$\operatorname{Att}_R H^c_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \operatorname{Supp}_R M : \operatorname{Ann}_R H^c_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p}\}.$$

Recall that a subcategory $\mathscr{M}$ of $\mathscr{C}(R)$ is said to be a *Serre category* if in any short exact sequence of R-modules and R-homomorphisms, the middle module is in $\mathscr{M}$ iff two other modules are in $\mathscr{M}$.

Lemma 2.2. *(See [4, Theorem 2.3]) Let $\mathfrak{a}$ be a proper ideal of a Noetherian ring R and let M and N be finitely generated R-modules such that $\operatorname{Supp} M \subseteq \operatorname{Supp} N$. Let $\mathscr{M}$ be a Serre category of R-modules and let $n \geq 0$ be an integer such that for each $i \geq n$ the local cohomology module $H^i_{\mathfrak{a}}(N)$ belongs to $\mathscr{M}$. Then for each $i \geq n$ the local cohomology module $H^i_{\mathfrak{a}}(M)$ belongs to $\mathscr{M}$.*

Now, we are ready to state and prove the first main result of this paper.

Theorem 2.3. *Let $\mathfrak{a}$ be an ideal of a Noetherian complete local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module with $\operatorname{cd}(\mathfrak{a}, M) = c \geq 1$ such that the R-module $H^c_{\mathfrak{a}}(M)$ is Artinian. Then,*

$$\operatorname{Att}_R H^c_{\mathfrak{a}}(M) \subseteq \operatorname{mAss}_R M \cup \{\mathfrak{p} \in \operatorname{Supp}_R M : \operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p} = \operatorname{Ann}_R H^c_{\mathfrak{a}}(R/\mathfrak{p})\}.$$

Proof. Set $R' := R/\operatorname{Ann}_R M$ and $\mathfrak{a}' := \mathfrak{a} R'$. By *Independence Theorem*, for each $i \in \mathbb{N}_0$ we have $H^i_{\mathfrak{a}}(M) \simeq H^i_{\mathfrak{a}'}(M)$. Therefore, $\operatorname{cd}(\mathfrak{a}', M) = \operatorname{cd}(\mathfrak{a}, M) = c$ and the R'-module $H^c_{\mathfrak{a}'}(M)$ is Artinian. Also, it is clear that $\operatorname{Att}_{R'} H^c_{\mathfrak{a}'}(M) = \{\mathfrak{p} R' : \mathfrak{p} \in \operatorname{Att}_R H^c_{\mathfrak{a}}(M)\}$, $\operatorname{mAss}_{R'} M = \{\mathfrak{p} R' : \mathfrak{p} \in \operatorname{mAss}_R M\} = \operatorname{mAss}_{R'} R'$ and

$$\{P \in \operatorname{Supp}_{R'} M : \operatorname{Ann}_{R'} H^{c-1}_{\mathfrak{a}'}(R'/P) = P = \operatorname{Ann}_{R'} H^c_{\mathfrak{a}'}(R'/P)\} = \{\mathfrak{p} R' : \mathfrak{p} \in \Phi\},$$

where $\Phi := \{\mathfrak{p} \in \operatorname{Supp}_R M : \operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p} = \operatorname{Ann}_R H^c_{\mathfrak{a}}(R/\mathfrak{p})\}$. Therefore, by passing to the quotient ring R' and by replacing $\mathfrak{a}$ with $\mathfrak{a}'$, we can make the additional assumption that M is a finitely generated faithful R-module, i.e., $\operatorname{Ann}_R M = 0$. Then we have $\operatorname{Supp}_R M = \operatorname{Spec} R$ and $\operatorname{mAss}_R M = \operatorname{mAss}_R R$.

Let $\mathfrak{p} \in \big(\operatorname{Att}_R H^c_{\mathfrak{a}}(M) \setminus \operatorname{mAss}_R M\big)$. Then by Lemma 2.1, we have $\operatorname{Ann}_R H^c_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p}$. Therefore, we need only establish that $\operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p}$. Assume the opposite. Then from the relations

$$\mathfrak{p} \subseteq \operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(R/\mathfrak{p}) \text{ and } \operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(R/\mathfrak{p}) \neq \mathfrak{p},$$

it is concluded that $\mathfrak{p} \subset \operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(R/\mathfrak{p})$. Let $\mathfrak{b}$ be an ideal of R with $\operatorname{Supp} R/\mathfrak{b} = V(\mathfrak{p})$ and set $L := R/\mathfrak{b}$. Then L has a composition series

$$0 = L_0 \subset L_1 \subset \cdots \subset L_i \subset \cdots \subset L_t = L$$

where the successive quotients are of the form $R/\mathfrak{q}$ with $\mathfrak{q} \in V(\mathfrak{p})$.

By induction on $1 \le i \le t$, we show that $\operatorname{Ann} H^{c-1}_{\mathfrak{a}}(L_i) \not\subseteq \mathfrak{p}$. Since $L_1 \simeq L_1/L_0 \simeq R/\mathfrak{q}_1$ for some $\mathfrak{q}_1 \in V(\mathfrak{p})$ and $\mathfrak{q}_1 \subseteq \operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(R/\mathfrak{q}_1) = \operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(L_1)$, so using the hypothesis $\mathfrak{p} \subset \operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(R/\mathfrak{p})$, we can deduce that $\operatorname{Ann} H^{c-1}_{\mathfrak{a}}(L_1) \not\subseteq \mathfrak{p}$. Thus, the assertion holds for $i = 1$.

Suppose, inductively, that $1 < i \le t$ and the result has been proved for smaller values of i. Then by the inductive hypothesis we have $\operatorname{Ann} H^{c-1}_{\mathfrak{a}}(L_{i-1}) \not\subseteq \mathfrak{p}$. Moreover, we have $L_i/L_{i-1} \simeq R/\mathfrak{q}_2$ for some $\mathfrak{q}_2 \in V(\mathfrak{p})$. Thus, by the same argument as for the case $i = 1$, one has $\operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(L_i/L_{i-1}) \not\subseteq \mathfrak{p}$. The short exact sequence

$$0 \longrightarrow L_{i-1} \longrightarrow L_i \longrightarrow L_i/L_{i-1} \longrightarrow 0,$$

induces the following exact sequence

$$H^{c-1}_{\mathfrak{a}}(L_{i-1}) \longrightarrow H^{c-1}_{\mathfrak{a}}(L_i) \longrightarrow H^{c-1}_{\mathfrak{a}}(L_i/L_{i-1}),$$

which shows that $(\operatorname{Ann} H^{c-1}_{\mathfrak{a}}(L_{i-1}))(\operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(L_i/L_{i-1})) \subseteq \operatorname{Ann} H^{c-1}_{\mathfrak{a}}(L_i)$. Therefore, $\operatorname{Ann} H^{c-1}_{\mathfrak{a}}(L_i) \not\subseteq \mathfrak{p}$. This completes the inductive step. Hence, $\operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(R/\mathfrak{b}) \not\subseteq \mathfrak{p}$. Let $D(-) = \operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$ denote the Matlis dual functor. Then, for every ideal $\mathfrak{b}$ of R with $\operatorname{Supp} R/\mathfrak{b} = V(\mathfrak{p})$ we have $\operatorname{Ann}_R D(H^{c-1}_{\mathfrak{a}}(R/\mathfrak{b})) \not\subseteq \mathfrak{p}$, i.e., $\mathfrak{p} \notin \operatorname{Supp}_R D(H^{c-1}_{\mathfrak{a}}(R/\mathfrak{b}))$. Hence, $(D(H^{c-1}_{\mathfrak{a}}(R/\mathfrak{b})))_{\mathfrak{p}} = 0$, for every ideal $\mathfrak{b}$ of R with $\operatorname{Supp} R/\mathfrak{b} = V(\mathfrak{p})$.

By Lemma 2.2, the R-module $H^c_{\mathfrak{a}}(R/\mathfrak{p})$ is Artinian since $\operatorname{Supp}_R R/\mathfrak{p} \subseteq \operatorname{Supp}_R M$ and the R-modules $H^i_{\mathfrak{a}}(M)$ are Artinian for all integers $i \ge c$. As R is a complete ring and

$H^c_{\mathfrak a}(R/\mathfrak p)$ ia Artinian, it follows that $N := D(H^c_{\mathfrak a}(R/\mathfrak p))$ is a finitely generated R-module with $\operatorname{Ann}_R N = \operatorname{Ann}_R H^c_{\mathfrak a}(R/\mathfrak p) = \mathfrak p$. Therefore, $\mathfrak p \in \operatorname{mAss}_R N$ and $0 < \ell_{R_{\mathfrak p}}(N_{\mathfrak p}) < \infty$. Set $k := \ell_{R_{\mathfrak p}}(N_{\mathfrak p})$. By induction on $n := \ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak b R_{\mathfrak p})$, for each ideal $\mathfrak b$ of R with $\operatorname{Supp}_R R/\mathfrak b = V(\mathfrak p)$, we prove that $\ell_{R_{\mathfrak p}}(D(H^c_{\mathfrak a}(R/\mathfrak b))_{\mathfrak p}) = k\ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak b R_{\mathfrak p}) = kn$.

Suppose that $n = \ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak b R_{\mathfrak p}) = 1$. Then we have $\mathfrak p \in \operatorname{mAss}_R R/\mathfrak b$. So, there is a short exact sequence of R-modules

$$0 \longrightarrow R/\mathfrak p \longrightarrow R/\mathfrak b \longrightarrow R/\mathfrak c \longrightarrow 0, \quad (2.3.1)$$

where $\mathfrak c$ is an ideal such that $\ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak c R_{\mathfrak p}) = \ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak b R_{\mathfrak p}) - \ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak p R_{\mathfrak p}) = 1 - 1 = 0$. Thus, $\mathfrak c \nsubseteq \mathfrak p$ and $\operatorname{Supp}_R R/\mathfrak c \subseteq \big(V(\mathfrak p) \setminus \{\mathfrak p\}\big)$. Therefore, $\mathfrak p \notin \operatorname{Supp}_R D(H^{c-1}_{\mathfrak a}(R/\mathfrak c))$ and $\mathfrak p \notin \operatorname{Supp}_R D(H^c_{\mathfrak a}(R/\mathfrak c))$. Also, the exact sequence (2.3.1) yields the following exact sequence

$$H^{c-1}_{\mathfrak a}(R/\mathfrak c) \longrightarrow H^c_{\mathfrak a}(R/\mathfrak p) \longrightarrow H^c_{\mathfrak a}(R/\mathfrak b) \longrightarrow H^c_{\mathfrak a}(R/\mathfrak c),$$

and from this exact sequence we get the following exact sequence

$$D(H^c_{\mathfrak a}(R/\mathfrak c)) \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak b)) \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak p)) \longrightarrow D(H^{c-1}_{\mathfrak a}(R/\mathfrak c)).$$

Furthermore, this exact sequence induces the following exact sequence

$$D(H^c_{\mathfrak a}(R/\mathfrak c))_{\mathfrak p} \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak b))_{\mathfrak p} \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak p))_{\mathfrak p} \longrightarrow D(H^{c-1}_{\mathfrak a}(R/\mathfrak c))_{\mathfrak p},$$

and from this exact sequence we obtain the isomorphism of $R_{\mathfrak p}$-modules

$$D(H^c_{\mathfrak a}(R/\mathfrak b))_{\mathfrak p} \simeq D(H^c_{\mathfrak a}(R/\mathfrak p))_{\mathfrak p}.$$

Consequently, we see that $\ell_{R_{\mathfrak p}}(D(H^c_{\mathfrak a}(R/\mathfrak b))_{\mathfrak p}) = \ell_{R_{\mathfrak p}}(D(H^c_{\mathfrak a}(R/\mathfrak p))_{\mathfrak p}) = \ell_{R_{\mathfrak p}}(N_{\mathfrak p}) = k$. So, the assertion holds for $n = 1$.

Suppose, inductively, that $n = \ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak b R_{\mathfrak p}) > 1$ and the result has been proved for smaller values of n. Then there is a short exact sequence of R-modules

$$0 \longrightarrow R/\mathfrak p \longrightarrow R/\mathfrak b \longrightarrow R/\mathfrak b_1 \longrightarrow 0, \quad (2.3.2)$$

where $\mathfrak b_1$ is an ideal of R such that $\operatorname{Supp}_R R/\mathfrak b_1 = V(\mathfrak p)$ and $\ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak b_1 R_{\mathfrak p}) = n-1$. Then, by inductive assumption, we have

$$\ell_{R_{\mathfrak p}}(D(H^c_{\mathfrak a}(R/\mathfrak b_1))_{\mathfrak p}) = k\ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak b_1 R_{\mathfrak p}) = k(n-1).$$

Since $\operatorname{Supp} R/\mathfrak b_1 \subseteq \operatorname{Supp} M$, by Lemma 2.1, one has $\operatorname{cd}(\mathfrak a, R/\mathfrak b_1) \le \operatorname{cd}(\mathfrak a, M) = c$. Thus, $H^{c+1}_{\mathfrak a}(R/\mathfrak b_1) = 0$. Therefore, using the fact that $H^{c+1}_{\mathfrak a}(R/\mathfrak p) = 0$, from the exact sequence (2.3.2) we get the following exact sequence

$$H^{c-1}_{\mathfrak a}(R/\mathfrak b_1) \longrightarrow H^c_{\mathfrak a}(R/\mathfrak p) \longrightarrow H^c_{\mathfrak a}(R/\mathfrak b) \longrightarrow H^c_{\mathfrak a}(R/\mathfrak b_1) \longrightarrow 0,$$

and from this exact sequence we get the following exact sequence

$$0 \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak b_1)) \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak b)) \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak p)) \longrightarrow D(H^{c-1}_{\mathfrak a}(R/\mathfrak b_1)).$$

Furthermore, this exact sequence induces the following exact sequence of $R_{\mathfrak p}$-modules

$$0 \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak b_1))_{\mathfrak p} \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak b))_{\mathfrak p} \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak p))_{\mathfrak p} \longrightarrow D(H^{c-1}_{\mathfrak a}(R/\mathfrak b_1))_{\mathfrak p}.$$

Since $\operatorname{Ann}_R D(H^{c-1}_{\mathfrak a}(R/\mathfrak b_1)) \not\subseteq \mathfrak p$, so the last $R_{\mathfrak p}$-module of this exact sequence is zero. Consequently, we have the following short exact sequence of $R_{\mathfrak p}$-modules

$$0 \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak b_1))_{\mathfrak p} \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak b))_{\mathfrak p} \longrightarrow D(H^c_{\mathfrak a}(R/\mathfrak p))_{\mathfrak p} \longrightarrow 0.$$

Therefore, we see that

$$\ell_{R_{\mathfrak p}}(D(H^c_{\mathfrak a}(R/\mathfrak b))_{\mathfrak p}) = \ell_{R_{\mathfrak p}}(D(H^c_{\mathfrak a}(R/\mathfrak p))_{\mathfrak p}) + \ell_{R_{\mathfrak p}}(D(H^c_{\mathfrak a}(R/\mathfrak b_1))_{\mathfrak p}) = k + k(n-1) = kn.$$

This completes the inductive step.

Since $\operatorname{Supp} M = \operatorname{Spec} R$, $\operatorname{cd}(\mathfrak a, M) = c$ and the R-module $H^c_{\mathfrak a}(M)$ is Artinian, so by Lemma 2.2, we see that the R-module $H^c_{\mathfrak a}(R)$ is Artinian and $\operatorname{cd}(\mathfrak a, R) = c$. Thus, the descending chain of the submodules of $H^c_{\mathfrak a}(R)$,

$$\mathfrak p\, H^c_{\mathfrak a}(R) \supseteq \mathfrak p^2\, H^c_{\mathfrak a}(R) \supseteq \mathfrak p^3\, H^c_{\mathfrak a}(R) \supseteq \cdots,$$

implies that $\mathfrak p^t\, H^c_{\mathfrak a}(R) = \mathfrak p^{t+1}\, H^c_{\mathfrak a}(R)$, for some $t \in \mathbb N$. So, using [8, Exercise 6.1.8], we get

$$\begin{aligned} H^c_{\mathfrak a}(R/\mathfrak p^t) &\simeq H^c_{\mathfrak a}(R) \otimes_R R/\mathfrak p^t \\ &\simeq H^c_{\mathfrak a}(R)/\mathfrak p^t\, H^c_{\mathfrak a}(R) \\ &= H^c_{\mathfrak a}(R)/\mathfrak p^{t+1}\, H^c_{\mathfrak a}(R) \\ &\simeq H^c_{\mathfrak a}(R) \otimes_R R/\mathfrak p^{t+1} \\ &\simeq H^c_{\mathfrak a}(R/\mathfrak p^{t+1}). \end{aligned}$$

Hence,

$$k\ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak p^t R_{\mathfrak p}) = \ell_{R_{\mathfrak p}}(D(H^c_{\mathfrak a}(R/\mathfrak p^t))_{\mathfrak p}) = \ell_{R_{\mathfrak p}}(D(H^c_{\mathfrak a}(R/\mathfrak p^{t+1}))_{\mathfrak p}) = k\ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak p^{t+1} R_{\mathfrak p}).$$

Consequently, $\ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak p^t R_{\mathfrak p}) = \ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak p^{t+1} R_{\mathfrak p})$. On the other hand, the exact sequence

$$0 \longrightarrow \mathfrak p^t R_{\mathfrak p}/\mathfrak p^{t+1} R_{\mathfrak p} \longrightarrow R_{\mathfrak p}/\mathfrak p^{t+1} R_{\mathfrak p} \longrightarrow R_{\mathfrak p}/\mathfrak p^t R_{\mathfrak p} \longrightarrow 0,$$

shows that

$$\ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak p^{t+1} R_{\mathfrak p}) = \ell_{R_{\mathfrak p}}(\mathfrak p^t R_{\mathfrak p}/\mathfrak p^{t+1} R_{\mathfrak p}) + \ell_{R_{\mathfrak p}}(R_{\mathfrak p}/\mathfrak p^t R_{\mathfrak p}),$$

whence we deduce that $\ell_{R_{\mathfrak p}}(\mathfrak p^t R_{\mathfrak p}/\mathfrak p^{t+1} R_{\mathfrak p}) = 0$. So, $\mathfrak p^t R_{\mathfrak p} = \mathfrak p^{t+1} R_{\mathfrak p}$. Thus, $\mathfrak p^t R_{\mathfrak p} = 0$, by *Nakayama's Lemma*. Therefore, $\dim_{R_{\mathfrak p}} R_{\mathfrak p} = 0$, i.e., $\mathfrak p \in \operatorname{mAss}_R R = \operatorname{mAss}_R M$. This is the desired contradiction. □

Proposition 2.4. *Let $\mathfrak a$ be an ideal of a Noetherian complete local ring $(R, \mathfrak m)$ and M be a finitely generated R-module with $\operatorname{cd}(\mathfrak a, M) = c \geq 1$ such that the R-module $H^c_{\mathfrak a}(M)$ is Artinian. Then the following statements are equivalent:*

1. $\operatorname{Att}_R H^c_{\mathfrak{a}}(M) \subseteq \operatorname{mAss}_R M$.
2. *If* $\mathfrak{p} \in \operatorname{Supp}_R M$ *and* $\operatorname{Ann}_R H^c_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p} = \operatorname{Ann}_R H^{c-1}_{\mathfrak{a}}(R/\mathfrak{p})$, *then* $\mathfrak{p} \in \operatorname{mAss}_R M$.
3. $H^c_{\mathfrak{a}}(R/\mathfrak{p}) = 0$ *or* $H^{c-1}_{\mathfrak{a}}(R/\mathfrak{p}) = 0$, *for each* $\mathfrak{p} \in \big((\operatorname{Supp}_R M) \setminus (\operatorname{mAss}_R M)\big)$.
4. $H^c_{\mathfrak{a}}(R/\mathfrak{p}) = 0$, *for each* $\mathfrak{p} \in \big((\operatorname{Supp}_R M) \setminus (\operatorname{mAss}_R M)\big)$.

Proof. (1)$\Longleftrightarrow$(2) The assertion follows from Lemma 2.1 and Theorem 2.3.

(1)$\Longrightarrow$(4) This implication follows from Lemma 2.1.

(4)$\Longrightarrow$(3) This assertion is trivial.

(3)$\Longrightarrow$(1) This conclusion follows from Lemma 2.1 and Theorem 2.3. ∎

The following well known Lemmas will be useful in the proof of Theorem 2.7.

Lemma 2.5. *(See [24, Corollary 3.5]) Let R be a Noetherian ring and $\mathfrak{a} \subseteq \mathfrak{b}$ be two ideals of R. Let K be an R-module such that* $\operatorname{cd}(\mathfrak{a}, K) = t \geq 2$. *Then* $\operatorname{Tor}^R_i(R/\mathfrak{b}, H^t_{\mathfrak{a}}(K)) = 0$, *for* $i = 0, 1$.

Lemma 2.6. *(See [15, p. 3198]) Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and M be a finitely generated R-module. Then*

$$\operatorname{Supp} M/\mathfrak{a} M = \operatorname{Supp} \bigoplus_{i \in \mathbb{N}_0} H^i_{\mathfrak{a}}(M).$$

In particular, $\dim M/\mathfrak{a} M = \dim_R \bigoplus_{i \in \mathbb{N}_0} H^i_{\mathfrak{a}}(M)$.

Lemma 2.7. *(See [11, Proposition 2]) Let $\mathfrak{a}$, $\mathfrak{b}$ be two ideals of a Noetherian ring R and M be an R-module with* $\mathfrak{b} M = 0$ *and* $\operatorname{Supp} M \subseteq V(\mathfrak{a})$. *Then M is $\mathfrak{a}$-cofinite (as an R-module) if and only if M is $(\mathfrak{a}+\mathfrak{b})/\mathfrak{b}$-cofinite (as an $R/\mathfrak{b}$-module).*

Lemma 2.8. *(See [6, Corollary 2.7]) Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and M be a finitely generated R-module with* $\dim_R M/\mathfrak{a} M \leq 1$. *Then the R-modules $H^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite for all $i \in \mathbb{N}_0$.*

Lemma 2.9. *(See [23, Proposition 5.1]) Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and M be a finitely generated R-module with* $\dim_R M = d \geq 0$. *Then the R-module $H^d_{\mathfrak{a}}(M)$ is Artinian and $\mathfrak{a}$-cofinite.*

Lemma 2.10. *(See [23, Proposition 3.11]) Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and M be a finitely generated R-module. If $j \in \mathbb{N}_0$ is an integer such that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i \in \mathbb{N}_0$ with $i \neq j$, then this is the case also when $i = j$.*

Lemma 2.11. *(See [2, Theorem 2.1]) Let $\mathfrak{a}$ be an ideal of a Noetherian complete local ring R and M be a non-zero finitely generated R-module with* $\operatorname{cd}(\mathfrak{a}, M) = c \geq 0$ *such that the R-module $H^c_{\mathfrak{a}}(M)$ is Artinian and $\mathfrak{a}$-cofinite. Then*

$$\operatorname{Att}_R H^c_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \operatorname{mAss}_R M : \dim_R R/\mathfrak{p} = c \text{ and } \sqrt{\mathfrak{a}+\mathfrak{p}} = \mathfrak{m}\}.$$

Now, we are ready to establish our second main result.

Theorem 2.12. *Let $\mathfrak{a}$ be an ideal of a Noetherian complete local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module with $\operatorname{cd}(\mathfrak{a}, M) = c \geq 1$ such that the R-module $H^c_{\mathfrak{a}}(M)$ is Artinian. Assume that $\mathfrak{p}$ is an attached prime ideal of the R-module $H^c_{\mathfrak{a}}(M)$. Then the following statements are equivalent:*

1. $\dim_R R/\mathfrak{p} = c$
2. *The R-module $H^c_{\mathfrak{a}}(R/\mathfrak{p})$ is $\mathfrak{a}$-cofinite.*
3. $\mathfrak{p} \in \operatorname{mAss}_R M$, $\sqrt{\mathfrak{a}+\mathfrak{p}} = \mathfrak{m}$ *and* $\dim R/\mathfrak{p} = c$.
4. $\dim_R R/(\mathfrak{a}+\mathfrak{p}) \leq 1$.
5. $\dim_R \bigoplus_{i\in\mathbb{N}_0} H^i_{\mathfrak{a}}(R/\mathfrak{p}) = 0$.
6. $\dim_R \bigoplus_{i\geq 2} H^i_{\mathfrak{a}}(R/\mathfrak{p}) = 0$.
7. $\dim_R \bigoplus_{i\geq 2} H^i_{\mathfrak{a}}(R/\mathfrak{p}) \leq 1$.

Proof. The conclusions (3)$\Longrightarrow$(1), (3)$\Longrightarrow$(5), (3)$\Longrightarrow$(4), (5)$\Longrightarrow$(6), (6)$\Longrightarrow$(7) are obvious.

(1)$\Longrightarrow$(2) This implication follows from Lemma 2.9.

(2)$\Longrightarrow$(3) By Lemma 2.7 and the proof of Theorem 2.3, without loss of generality we may assume that $\operatorname{Ann}_R M = 0$. Then by Lemmas 2.1 and 2.2, we have $\operatorname{Ann}_R H^c_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p}$, $\operatorname{cd}(\mathfrak{a}, R) = c$. Therefore, the assertion follows from the proof of [3, Lemma 2.3].

(4)$\Longrightarrow$(2) The assertion follows from Lemma 2.8.

(7)$\Longrightarrow$(4) If $c = 1$, then by Lemma 2.10, the Artinian R-module $H^1_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. So, $\sqrt{\mathfrak{a}+\mathfrak{p}} = \mathfrak{m}$, by Lemma 2.11. Hence, $\dim_R R/(\mathfrak{a}+\mathfrak{p}) = 0 \leq 1$. Thus, we may assume that $c \geq 2$. Since $\mathfrak{p} \in \operatorname{Att}_R H^c_{\mathfrak{a}}(M)$, so $\operatorname{Ann}_R H^c_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p}$, by Lemma 2.1. Furthermore, by Lemma 2.2, $H^c_{\mathfrak{a}}(R/\mathfrak{p})$ is Artinian and $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = c$ since $\operatorname{Supp}_R R/\mathfrak{p} \subseteq \operatorname{Supp}_R M$. Set $N := D(H^c_{\mathfrak{a}}(R/\mathfrak{p}))$, where $D(-)$ denotes the Matlis dual functor. Then, N is a finitely generated R-module with $\operatorname{Ann}_R N = \operatorname{Ann}_R H^c_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p}$. In addition, it follows from Lemma 2.5 and adjointness that

$$\operatorname{Ext}^i_R(R/\mathfrak{a}, N) \simeq D(\operatorname{Tor}^R_i(R/\mathfrak{a}, H^c_{\mathfrak{a}}(R))) = 0, \text{ for } i = 0, 1.$$

This means that $\operatorname{grade}(\mathfrak{a}, N) \geq 2$ and hence $H^i_{\mathfrak{a}}(N) = 0$ for $i = 0, 1$. On the other hand, it follows from from Lemma 2.2 that $\dim_R \bigoplus_{i\geq 2} H^i_{\mathfrak{a}}(N) \leq 1$ since $\operatorname{Supp} N = \operatorname{Supp} R/\mathfrak{p}$. Therefore, $\dim_R H^i_{\mathfrak{a}}(N) \leq 1$ for all $i \in \mathbb{N}_0$. So, by Lemma 2.6, we have

$$\dim_R R/(\mathfrak{a}+\mathfrak{p}) = \dim_R N/\mathfrak{a}N = \dim_R \bigoplus_{i\in\mathbb{N}_0} H^i_{\mathfrak{a}}(N) \leq 1.$$

∎

Corollary 2.13. *Let $\mathfrak{a}$ be an ideal of a Noetherian complete local ring $(R,\mathfrak{m})$ and M be a finitely generated R-module with $\operatorname{cd}(\mathfrak{a},M)=c\geq 1$ such that the R-module $H^c_{\mathfrak{a}}(M)$ is Artinian. Let $\mathfrak{p}\in \operatorname{Att}_R H^c_{\mathfrak{a}}(M)$ and $\dim_R R/\mathfrak{p}=c+1$. Then $\mathfrak{p}$ is a maximal element of the set $\operatorname{Att}_R H^c_{\mathfrak{a}}(M)$ with respect to inclusion.*

Proof. Assume the opposite. Then there is an element $\mathfrak{q}\in \operatorname{Att}_R H^c_{\mathfrak{a}}(M)$ such that $\mathfrak{p}\subset\mathfrak{q}$. Then by Lemma 2.1, $\mathfrak{q}\in \operatorname{Supp}_R M$, $\dim_R R/\mathfrak{q}\leq \dim_R R/\mathfrak{p}-1=c$ and $H^c_{\mathfrak{a}}(R/\mathfrak{q})\neq 0$. So, by *Grothendieck's Vanishing Theorem*, we have $\dim_R R/\mathfrak{q}\geq c$. Therefore, $\dim_R R/\mathfrak{q}=c$ and hence $\mathfrak{q}\in \operatorname{mAss}_R M$, by Theorem 2.12. But, by Lemma 2.1 we have $\mathfrak{p}\in \operatorname{Supp} M$. Since $\mathfrak{p}\subset\mathfrak{q}$, so $\mathfrak{q}\notin \operatorname{mAss}_R M$. This is the desired contradiction. ∎

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Payame Noor University, 19395-3697 Tehran, Iran
E-mail: `gh_pirmohammadi@pnu.ac.ir`