Bull. Math. Soc. Sci. Math. Roumanie
Tome 69 (117), No. 2, 2026, 119–147

Lattice paths related to descents and inverse descents in hyperoctahedral groups

by
Xiaoqin Gao[(1)], Frank Z. K. Li[(2)], Lingli Wan[(3)], Jane Y. X. Yang[(4)]

Abstract

The elements in the hyperoctahedral group $\mathfrak{B}_n$ can be treated as signed permutations with the natural order $\cdots < -2 < -1 < 0 < 1 < 2 < \cdots$, or as colored permutations with the r-order $-1 <_r -2 <_r \cdots <_r 0 <_r 1 <_r 2 <_r \cdots$. For any $\pi \in \mathfrak{B}_n$, let $\mathrm{des}^B(\pi)$ and $\mathrm{ides}^B(\pi)$ be the number of descents and inverse descents in π under the natural order, respectively, and let $\mathrm{des}_B(\pi)$ and $\mathrm{ides}_B(\pi)$ be the number of descents and inverse descents in π under the r-order, respectively. Visontai algebraically studied the joint distribution of $(\mathrm{des}^B, \mathrm{ides}^B)$. In this paper, we introduce a new class of lattice paths on permutation grids that give a combinatorial proof for the recurrence formula of $(\mathrm{des}^B, \mathrm{ides}^B)$ arising from the work of Visontai. And by modifying certain combinatorial structures from the natural order to the r-order, we acquire a recurrence formula of the joint distribution of $(\mathrm{des}_B, \mathrm{ides}_B)$. As a conclusion, we find that $(\mathrm{des}^B, \mathrm{ides}^B)$ and $(\mathrm{des}_B, \mathrm{ides}_B)$ are equidistributed over the hyperoctahedral group.

Key Words: Hyperoctahedral group, inverse descents, signed permutation grids, recurrence formulas.
2020 Mathematics Subject Classification: 05A05, 05A19, 05E16.

1 Introduction

The main purpose of this paper is to introduce a new class of lattice paths that lead us to find combinatorial approaches to prove recurrence formulas related to the joint distribution of descents and inverse descents in hyperoctahedral groups. For any positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and $\Omega_n = \{0, 1, -1, 2, -2, \ldots, n, -n\}$. Following the notations in [1, Section 2], we denote by $\mathfrak{B}_n$ the set of bijections $\pi : \Omega_n \to \Omega_n$ such that $\pi(a) = b$ implying $\pi(-a) = -b$ for any $a \in \Omega_n$. By this restriction, we always have $\pi(0) = 0$ and we can use $\pi = \pi_1\pi_2 \cdots \pi_n$ with $\pi_i = \pi(i)$ for $i \in [n]$ to indicate elements in $\mathfrak{B}_n$, where $\overline{a}$ stands for $-a$ for $a \in [n]$. For example, we write $\overline{3}16\overline{5}24 \in \mathfrak{B}_6$.

The set $\mathfrak{B}_n$ is widely known as the *hyperoctahedral group*, the *Coxeter group* with type B (see [4, Section 8.1]), which can be regarded as the set of signed permutations. In addition, the hyperoctahedral group $\mathfrak{B}_n$ can be also viewed as the *wreath product* $\mathbb{Z}_r \wr \mathfrak{S}_n$ with $r = 2$, where $\mathfrak{S}_n$ is the symmetric group, and $\mathbb{Z}_r$ is the cyclic group of order r. The wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$ can be realized as the r-colored permutation group $\mathfrak{S}_{n,r}$, which allows us to treat elements in $\mathfrak{B}_n$ as 2-colored permutations. We refer the reader to [3, Section 2.1.7] and [11, Section 1.2] to check more information on wreath product and colored permutations.

Since the hyperoctahedral group $\mathfrak{B}_n$ can be realized by signed permutations or colored permutations, it is necessary to choose a suitable total order on $\mathbb{Z}$ when studying statistics over $\mathfrak{B}_n$. The natural order

$$\cdots < -2 < -1 < 0 < 1 < 2 < \cdots \tag{1}$$

is used when considering $\mathfrak{B}_n$ as the Coxeter group of type B. And the r-order

$$-1 <_r -2 <_r \cdots <_r 0 <_r 1 <_r 2 <_r \cdots \tag{2}$$

is used when considering $\mathfrak{B}_n$ as the 2-colored permutation group, where r stands for the lexicographic order on the set $\mathbb{Z}_{>0}\{-,+\}$ (see [1, Equation (2.1)]). For $\pi \in \mathfrak{B}_n$ and $0 \leq i \leq n-1$, we set $\pi_0 = 0$ and define

$$\mathrm{Des}^B(\pi) = \{i \in [0, n-1] \mid \pi_i > \pi_{i+1}\} \quad \text{and} \quad \mathrm{Des}_B(\pi) = \{i \in [0, n-1] \mid \pi_i >_r \pi_{i+1}\},$$

The indices i in $\mathrm{Des}^B(\pi)$ and $\mathrm{Des}_B(\pi)$ are called *$\overline{B}$-descents* and *$\underline{B}$-descents* in π, respectively. Without causing confusion, we refer to $\overline{B}$-descents and $\underline{B}$-descents both as *descents* in π. Let

$$\mathrm{des}^B(\pi) = |\,\mathrm{Des}^B(\pi)| \quad \text{and} \quad \mathrm{des}_B(\pi) = |\,\mathrm{Des}_B(\pi)|.$$

The statistics des^B and des_B record the number of descents in signed permutations (see [14, Section 13.1]) and colored permutations (see [17, Section 2]), respectively. Actually, in [2, Page 218], Adin et al. remark that des^B and des_B are equally distributed over $\mathfrak{B}_n$. For any positive integer n, the $\mathfrak{B}_n$-Eulerian polynomial $B_n(t)$ is defined as

$$B_n(t) = \sum_{\pi \in \mathfrak{B}_n} t^{\mathrm{des}^B(\pi)} = \sum_{\pi \in \mathfrak{B}_n} t^{\mathrm{des}_B(\pi)}. \tag{3}$$

As with the classical Eulerian polynomial, the $\mathfrak{B}_n$-Eulerian polynomial is proved to be γ-positivity, where combinatorial interpretations for the γ-coefficients in $B_n(t)$ can be found in [3, Theorem 2.10], [7, Theorem 4.7] and [12, Proposition 4.15].

In this paper, we mainly focus on the joint distribution of descents and inverse descents on $\mathfrak{B}_n$. The indices i in the set

$$\mathrm{iDes}^B(\pi) = \{i \in [0, n-1] \mid \pi_i^{-1} > \pi_{i+1}^{-1}\}$$

are called *inverse $\overline{B}$-descents*, and in the set

$$\mathrm{iDes}_B(\pi) = \{i \in [0, n-1] \mid \pi_i^{-1} >_r \pi_{i+1}^{-1}\}$$

are called *inverse $\underline{B}$-descents*. The inverse $\overline{B}$-descents and the inverse $\underline{B}$-descents are both regarded as *inverse descents* in π. Moreover, they are also simply referred as *idescents* (see [15]). Denote by

$$\mathrm{ides}^B(\pi) = |\,\mathrm{iDes}^B(\pi)|, \qquad \mathrm{ides}_B(\pi) = |\,\mathrm{iDes}_B(\pi)|.$$

For $0 \leq i, j \leq n$, let

$$\overline{\mathcal{B}}_{n,i,j} = \{\pi \in \mathfrak{B}_n \mid \ \mathrm{des}^B(\pi) = i \text{ and } \mathrm{ides}^B(\pi) = j\}$$

and $\overline{b}_{n,i,j} = |\overline{\mathcal{B}}_{n,i,j}|$. The generating function of $\overline{b}_{n,i,j}$ is called as the type B two-side Eulerian polynomial (see [18, Section 4.1]):

$$\overline{B}_n(s,t) = \sum_{\pi \in \mathfrak{B}_n} s^{\mathrm{des}^B(\pi)} t^{\mathrm{ides}^B(\pi)} = \sum_{0 \le i,j \le n} \overline{b}_{n,i,j} s^i t^j. \tag{4}$$

The classical two-side Eulerian polynomial $A_n(s,t)$ is defined on the symmetric group $\mathfrak{S}_n$, where one can see [13, Page 167] for details. Thus the polynomial $\overline{B}_n(s,t)$ is a generalization of $A_n(s,t)$ from the Coxeter group of type A to type B. Starting with an identity of binomial coefficients given by Petersen to prove [13, Equation (9)], Visontai [18] derived a differential equation about $\overline{B}_n(s,t)$.

Theorem 1.1 ([18], Theorem 15)**.** *For $n \ge 2$,*

$$\begin{aligned} n\overline{B}_n(s,t) = {} & \left(2n^2 st - nst + n\right) \overline{B}_{n-1}(s,t) \\ & + (2nst(1-s) + s(1-s)(1-t)) \frac{\partial}{\partial s} \overline{B}_{n-1}(s,t) \\ & + (2nst(1-t) + t(1-s)(1-t)) \frac{\partial}{\partial t} \overline{B}_{n-1}(s,t) \\ & + 2st(1-s)(1-t) \frac{\partial^2}{\partial s \partial t} \overline{B}_{n-1}(s,t) \end{aligned} \tag{5}$$

with initial value $\overline{B}_1(s,t) = 1 + st$.

Comparing the coefficients of $\overline{B}_n(s,t)$ in the two sides of (5), a recurrence of $\overline{b}_{n,i,j}$ is obtained.

Theorem 1.2. *For $n \ge 2$ and $0 \le i,j \le n$, we have*

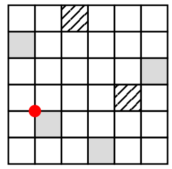
$$\begin{aligned} n\overline{b}_{n,i,j} = {} & (n + i + j + 2ij)\overline{b}_{n-1,i,j} \\ & + ((2n-1)i - (2i+1)(j-1))\overline{b}_{n-1,i,j-1} \\ & + ((2n-1)j - (2j+1)(i-1))\overline{b}_{n-1,i-1,j} \\ & + ((2n^2 - n) + 2(i-1)(j-1) + (1-2n)(i+j-2))\overline{b}_{n-1,i-1,j-1}, \end{aligned} \tag{6}$$

where $\overline{b}_{1,0,0} = \overline{b}_{1,1,1} = 1$, $\overline{b}_{1,0,1} = \overline{b}_{1,1,0} = 0$ and $\overline{b}_{n,i,j} = 0$ if $i < 0$ or $j < 0$.

The geometric representations of permutations [16, Section 1.5] such as matrices, grids, trees provide us with intuitive perspectives to study the properties of permutations. We can extend the definition of permutation grids to signed permutation grids by the following natural way.

Definition 1.3. *For any signed permutation $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{B}_n$, the* signed permutation grid *P_π of π is defined as an $n \times n$ grid with the $|\pi_i|$-th (from the left to the right) square in the i-th (from the top to the bottom) row filled in if $\pi_i > 0$, and filled by slashes if $\pi_i < 0$.*

For accuracy, we refer to $\langle i,j \rangle \in [n] \times [n]$ and $(i,j) \in [n+1] \times [n+1]$ as the square located in the i-th row and the j-th column, and the grid point intersected by the i-th horizontal line and the j-th vertical line, respectively. For $1 \le i \le n$, we define the sign of the square by calling $\langle i, -\pi_i \rangle$ as *negative square*, and $\langle i, \pi_i \rangle$ as *positive square*. For example, the signed permutation grid of $\pi = \overline{3}16\overline{5}24$ is given by



with a negative square $\langle 4,5\rangle$ and a positive square $\langle 3,6\rangle$, and the grid point $(5,2)$ addressed in red.

We introduce certain square pairs and lattice paths with respect to $\overline{B}$-descents and $\overline{B}$-idescents in permutation grids, which are described in detail in Section 2. Thus by characterizing how $\overline{B}$-descents and $\overline{B}$-idescents are geometrically reflected on the signed permutation grid, as well as how they change during the growth of an $n\times n$ grid to an $(n+1)\times(n+1)$ grid, we give a combinatorial interpretation of Theorem 1.2, which is one of the main results of this paper.

On the other hand, we come up to study descents and idescents in $\mathfrak{B}_n$ with r-order. For $0\le i,j\le n$, let

$$\underline{\mathcal{B}}_{n,i,j}=\{\pi\in\mathfrak{B}_n\mid\ \mathrm{des}_B(\pi)=i \text{ and } \mathrm{ides}_B(\pi)=j\},$$

and $\underline{b}_{n,i,j}=|\underline{\mathcal{B}}_{n,i,j}|$. Modifying the combinatorial objects related to descents and idescents in signed permutation grids from the natural order to the r-order, we derive the following theorem that gives a recursive formula of $\underline{b}_{n,i,j}$, which characterizes the joint distribution of $\underline{B}$-descents and $\underline{B}$-idescents over $\mathfrak{B}_n$, and is another main result of the paper.

Theorem 1.4. *For $n\ge 2$ and $0\le i,j\le n$, we have*

$$\begin{aligned}n\underline{b}_{n,i,j}=&(n+i+j+2ij)\underline{b}_{n-1,i,j}\\&+((2n-1)i-(2i+1)(j-1))\underline{b}_{n-1,i,j-1}\\&+((2n-1)j-(2j+1)(i-1))\underline{b}_{n-1,i-1,j}\\&+((2n^2-n)+2(i-1)(j-1)+(1-2n)(i+j-2))\underline{b}_{n-1,i-1,j-1},\end{aligned}\tag{7}$$

where $\underline{b}_{1,0,0}=\underline{b}_{1,1,1}=1$, $\underline{b}_{1,0,1}=\underline{b}_{1,1,0}=0$ and $\underline{b}_{n,i,j}=0$ if $i<0$ or $j<0$.

By Theorems 1.2 and 1.4, we find that $\underline{b}_{n,i,j}$ shares exactly the same recursive relation and initial values with $\overline{b}_{n,i,j}$. Hence, we see the joint statistics $(\mathrm{des}^B,\mathrm{ides}^B)$ and $(\mathrm{des}_B,\mathrm{ides}_B)$ are equally distributed over $\mathfrak{B}_n$, which is a refinement result on the equidistribution property of des^B and des_B over $\mathfrak{B}_n$.

Particularly, the type B two-side Eulerian polynomial (4) can be unified to the two-side $\mathfrak{B}_n$-Eulerian polynomial

$$B_n(s,t)=\sum_{\pi\in\mathfrak{B}_n}s^{\mathrm{des}^B(\pi)}t^{\mathrm{ides}^B(\pi)}=\sum_{\pi\in\mathfrak{B}_n}s^{\mathrm{des}_B(\pi)}t^{\mathrm{ides}_B(\pi)},$$

which is a generalization of $\mathfrak{B}_n$-Eulerian polynomial (3).

The rest of this paper is organized as follows. Section 2 is devoted to investigate geometric properties of $\overline{B}$-descents and $\overline{B}$-idescents under natural order in signed permutation grids, especially derive enumerative formulas for certain internal structures of grids. We combinatorially prove Theorem 1.2 in Section 3. By modifying the techniques with natural order to r-order, in Section 4, we study the properties of $\underline{B}$-descents and inverse $\underline{B}$-descents in grids, which is served as the preparation for combinatorial proof of Theorem 1.4. In Section 5, we make some further remarks.

2 Lattice paths related to $\overline{B}$-descents and $\overline{B}$-idescents

In this section, we begin by introducing the notations of relevant operations that are frequently used, then investigate the changes of $\overline{B}$-descents and $\overline{B}$-idescents during the growth processes from $n \times n$ signed permutation grids to $(n+1) \times (n+1)$ ones.

2.1 Inserting operations on signed permutation grids

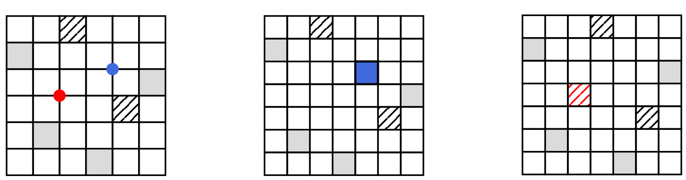
Let $\pi \in \mathfrak{B}_n$ and $i, j \in [n+1]$, then the *inserting operation* $\varphi_{(i,j)}$ is defined as $\varphi_{(i,j)}(\pi) = \sigma \in \mathfrak{B}_{n+1}$ such that for $k \in [n+1]$,

$$\sigma_k = \begin{cases} j, & k = i, \\ \mathrm{sgn}(\pi_{k-1})(|\pi_{k-1}|+1), & k > i \text{ and } |\pi_{k-1}| \geq j, \\ \pi_{k-1}, & k > i \text{ and } |\pi_{k-1}| < j, \\ \mathrm{sgn}(\pi_k)(|\pi_k|+1), & k < i \text{ and } |\pi_k| \geq j, \\ \pi_k, & k < i \text{ and } |\pi_k| < j, \end{cases}$$

and the *inserting operation* $\overline{\varphi}_{(i,j)}$ is defined as $\overline{\varphi}_{(i,j)}(\pi) = \sigma =\in \mathfrak{B}_{n+1}$ such that for $k \in [n+1]$,

$$\sigma_k = \begin{cases} \overline{j}, & k = i, \\ \mathrm{sgn}(\pi_{k-1})(|\pi_{k-1}|+1), & k > i \text{ and } |\pi_{k-1}| \geq j, \\ \pi_{k-1}, & k > i \text{ and } |\pi_{k-1}| < j, \\ \mathrm{sgn}(\pi_k)(|\pi_k|+1), & k < i \text{ and } |\pi_k| \geq j, \\ \pi_k, & k < i \text{ and } |\pi_k| < j, \end{cases}$$

where sgn is the signum function. From the graphic view, the operations $\varphi_{(i,j)}$ and $\overline{\varphi}_{(i,j)}$ inserting a positive square and a negative square, respectively, at the grid point (i, j), while preserve the relative positions of the original filled squares. For example, if $\pi = \overline{3}16\overline{5}24$, then the grids of $\varphi_{(3,5)}(\pi) = \overline{3}157\overline{6}24$ and $\overline{\varphi}_{(4,3)}(\pi) = \overline{4}17\overline{3}\overline{6}25$ are given as follows.



P_π $\qquad$ $P_{\varphi_{(3,5)}(\pi)}$ $\qquad$ $P_{\overline{\varphi}_{(4,3)}(\pi)}$

Conversely, for any signed permutation $\sigma \in \mathfrak{B}_{n+1}$ with a filled square $\langle i, j \rangle$ in its grid P_σ, by deleting the i-th row and the j-th column of squares from the grid, we obtain a new grid P_π. This process can be described as deleting the positive square $\langle i, j \rangle$ if $\sigma_i = j$, or the negative square $\langle i,j\rangle$ if $\sigma_i=\overline{j}$ from the grid P_σ, and denoted by the *deleting operations* $\varphi^{-1}_{\langle i,j\rangle}$ or $\overline{\varphi}^{-1}_{\langle i,j\rangle}$. More precisely, we have $\pi=\varphi^{-1}_{\langle i,j\rangle}(\sigma)$ with $\sigma_i=j$, or $\pi=\overline{\varphi}^{-1}_{\langle i,j\rangle}(\sigma)$ with $\sigma_i=\overline{j}$, such that for $k\in[n]$,

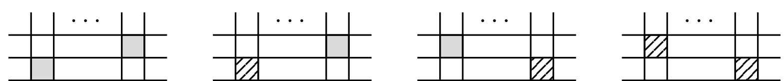
$$\pi_k=\begin{cases}\sigma_k, & k<i \text{ and } \sigma_k<j,\\ \operatorname{sgn}(\sigma_k)(|\sigma_k|-1), & k<i \text{ and } \sigma_k\geq j+1,\\ \sigma_{k+1}, & k\geq i \text{ and } \sigma_{k+1}<j,\\ \operatorname{sgn}(\sigma_{k+1})(|\sigma_{k+1}|-1), & k\geq i \text{ and } \sigma_{k+1}\geq j+1.\end{cases}$$

These two deleting operations can be roughly viewed as the inverse of the inserting operations $\varphi_{(i,j)}$ and $\overline{\varphi}_{(i,j)}$, since $\pi=\varphi^{-1}_{\langle i,j\rangle}(\sigma)$ if and only if $\sigma=\varphi_{(i,j)}(\pi)$, and $\pi=\overline{\varphi}^{-1}_{\langle i,j\rangle}(\sigma)$ if and only if $\sigma=\overline{\varphi}^{-1}_{\langle i,j\rangle}(\pi)$.

2.2 Descent types of square pairs and d-types of grid points

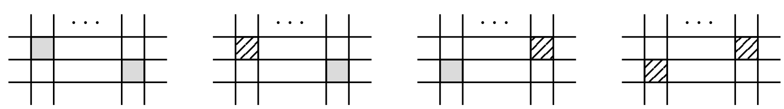
According to the different forms of $\overline{B}$-descents and $\overline{B}$-idescents, we group the pairs of filled squares in adjacent rows or adjacent columns into eight different types.

Since the filled squares in the signed permutation grid are positive or negative , there are four different pairs of filled squares in adjacent rows as *des-pairs* can form $\overline{B}$-descents, which are called *des*$^+_+$*-pair*, *des*$^-_+$*-pair*, *des*$^+_-$*-pair* and *des*$^-_-$*-pair* as follows.



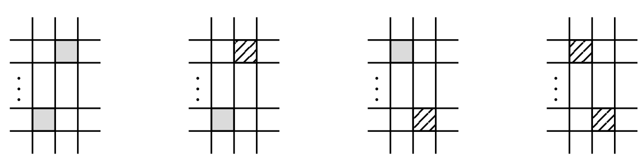
des$^+_+$-pair des$^-_+$-pair des$^+_-$-pair des$^-_-$-pair

For $\pi=\pi_1\pi_2\ldots\pi_n\in\mathfrak{B}_n$ with $\pi_0=0$, an index $i\in[0,n-1]$ is defined to be a $\overline{B}$*-ascent* if $\pi_i<\pi_{i+1}$, or a $\overline{B}$*-iascent* if $\pi^{-1}_i<\pi^{-1}_{i+1}$. Thus the other four pairs of filled squares in adjacent rows as *asc-pairs* form $\overline{B}$-ascents, which are called as *asc*$^+_+$*-pair*, *asc*$^-_+$*-pair*, *asc*$^+_-$*-pair* and *asc*$^-_-$*-pair*.

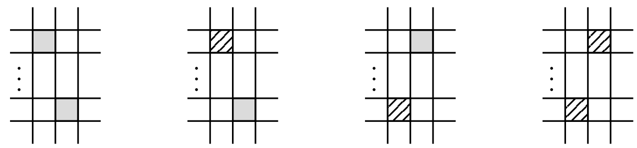


asc$^+_+$-pair asc$^-_+$-pair asc$^+_-$-pair asc$^-_-$-pair

It can be directly seen that among the eight different pairs of filled squares in adjacent columns, four of them as *ides-pairs* form $\overline{B}$-idescents, and the remanning four as *iasc-pairs* form $\overline{B}$-iascents.

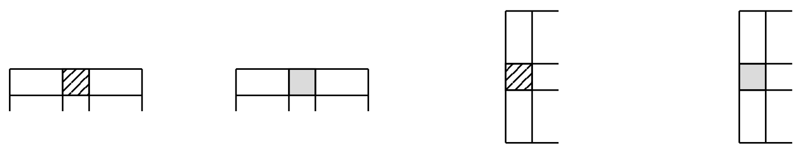


ides$^+_+$-pair ides$^-_+$-pair ides$^+_-$-pair ides$^-_-$-pair



iasc$^+_+$-pair iasc$^-_+$-pair iasc$^+_-$-pair iasc$^-_-$-pair

Furthermore, since $\pi_0 = 0$ for any $\pi \in \mathfrak{B}_n$, the filled squares in the first row or column of the grid P_π are classified into the following four different types as *des*$^-$*-square*, *asc*$^+$*-square*, *ides*$^-$*-square* and *iasc*$^+$*-square*.



des$^-$-square asc$^+$-square ides$^-$-square iasc$^+$-square

Definition 2.1. *Let $\pi \in \mathfrak{B}_n$ and P_π be its grid, then for $1 \le i, j \le n+1$, the grid point (i, j) is of d^+-*type *(p, q) and denoted by $d^+(i, j) = (p, q)$ if*

$$(\mathrm{des}^B(\varphi_{(i,j)}(\pi)), \mathrm{ides}^B(\varphi_{(i,j)}(\pi))) - (\mathrm{des}^B(\pi), \mathrm{ides}^B(\pi)) = (p, q),$$

*where p and q are called as the d_h^+-*type *and the d_v^+-*type*, and denoted by $d_h^+(i, j) = p$ and $d_v^+(i, j) = q$, respectively.*

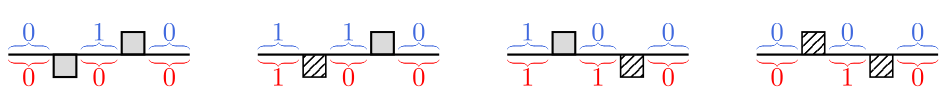
Definition 2.2. *Let $\pi \in \mathfrak{B}_n$ and P_π be its grid, then for $1 \le i, j \le n+1$, the grid point (i, j) is of d^--*type *(p, q) and denoted by $d^-(i, j) = (p, q)$ if*

$$(\mathrm{des}^B(\overline{\varphi}_{(i,j)}(\pi)), \mathrm{ides}^B(\overline{\varphi}_{(i,j)}(\pi))) - (\mathrm{des}^B(\pi), \mathrm{ides}^B(\pi)) = (p, q),$$

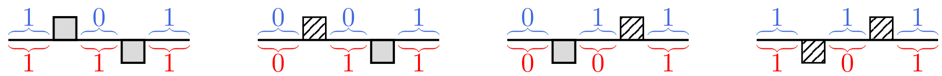
*where p and q are called as the d_h^--*type *and the d_v^--*type*, and denoted by $d_h^-(i, j) = p$ and $d_v^-(i, j) = q$, respectively.*

We call d^+-type and d^--type, d_h^+-type and d_h^--type, d_v^+-type and d_v^--type collectively as *d-type*, *d_h-type*, *d_v-type*, respectively, and use the superscripts $+$ and $-$ to distinguish the signs of the corresponding type. We see from the following propositions that the distribution of d_h-types is affected by des-pairs and asc-pairs, and the distribution of d_v-types is affected by ides-pairs and iasc-pairs.

Proposition 2.3. *The d_h-types of the grid points on the middle horizontal line of des-pairs and asc-pairs are indicated in the figure below, where the numbers above are d_h^+-types, and the numbers below are d_h^--types.*

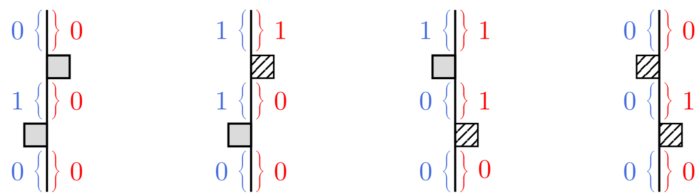


des$^+_+$*-pair* *des*$^-_+$*-pair* *des*$^+_-$*-pair* *des*$^-_-$*-pair*

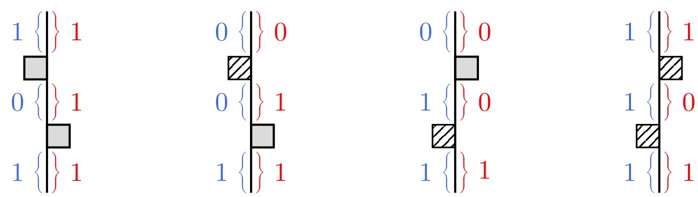


asc_+^+-pair *asc_+^--pair* *asc_-^+-pair* *asc_-^--pair*

Proposition 2.4. *The d_v-types of the grid points on the middle vertical line of ides-pairs and iasc-pairs are indicated in the figure below, where the numbers on the left are d_v^+-types, and the numbers on the right are d_v^--types.*



$ides_+^+$-pair *$ides_+^-$-pair* *$ides_-^+$-pair* *$ides_-^-$-pair*



$iasc_+^+$-pair *$iasc_+^-$-pair* *$iasc_-^+$-pair* *$iasc_-^-$-pair*

The validity of the above two propositions can be directly checked by taking operations $\varphi_{(i,j)}$ and $\overline{\varphi}_{(i,j)}$ at corresponding grid points (i, j). Using the same analysis with $\pi_0 = 0$, we obtain the next proposition that describes the distributions of d_h-types and d_v-types of grid points on the grid border.

Proposition 2.5. *The d_h-types of the grid points on the top and bottom boundaries of the grid are indicated in the figure below, where the numbers above are d_h^+-types, and the numbers below are d_h^--types.*

top boundary *bottom boundary*

And the d_v-types of the grid points on the left and right boundaries of the grid are indicated in the figure below, where the numbers on the left are d_v^+-types, and the numbers on the right are d_v^--types.

left boundary *right boundary*

Combining Propositions 2.3–2.5, we get all possible values of d-types, and the d-types of the four corners of filled squares.

Proposition 2.6. *Let $\pi \in \mathfrak{B}_n$ and P_π be its grid, then all possible d^+-types or d^--types of each grid point are only $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$.*

Proposition 2.7. *Let $\pi \in \mathfrak{B}_n$ and P_π be its grid, then for $1 \le i, j \le n$, we have*

$$d^+(i,j) = d^+(i+1,j+1) = (0,0) \quad \textit{and} \quad d^+(i,j+1) = d^+(i+1,j) = (1,1),$$

if $\langle i,j \rangle$ is a positive square in P_π, and

$$d^-(i,j) = d^-(i+1,j+1) = (1,1) \quad \textit{and} \quad d^-(i,j+1) = d^-(i+1,j) = (0,0).$$

if $\langle i,j \rangle$ is a negative square in P_π, as shown below.

d^+-types on a positive square *d^--types on a negative square*

Thanks to the above proposition, we arrive at more general conclusions that describe the circumstances under which the d_h-types or d_v-types on the same horizontal or vertical grid line change.

Proposition 2.8. *Let $\pi \in \mathfrak{B}_n$ and P_π be its grid, then for any two grid points (i,j_1) and (i,j_2) on the i-th horizontal grid line with $1 \le i \le n+1$ and $1 \le j_1 < j_2 \le n+1$, we have*

(a) $d_h^+(i,j_1) \ne d_h^+(i,j_2)$ if and only if there exists exactly one positive square $\langle i',j' \rangle$ satisfying $i-1 \le i' \le i$ and $j_1 \le j' < j_2$,

(b) $d_h^-(i,j_1) \ne d_h^-(i,j_2)$ if and only if there exists exactly one negative square $\langle i',j' \rangle$ satisfying $i-1 \le i' \le i$ and $j_1 \le j' < j_2$.

Proposition 2.9. *Let $\pi \in \mathfrak{B}_n$ and P_π be its grid, then for any two grid points (i_1,j) and (i_2,j) on the j-th vertical grid line with $1 \le j \le n+1$ and $1 \le i_1 < i_2 \le n+1$, we have*

(a) $d_v^+(i_1,j) \ne d_v^+(i_2,j)$ if and only if there exists exactly one positive square $\langle i',j' \rangle$ satisfying $j-1 \le j' \le j$ and $i_1 \le i' < i_2$,

(b) $d_v^-(i_1,j) \ne d_v^-(i_2,j)$ if and only if there exists exactly one negative square $\langle i',j' \rangle$ satisfying $j-1 \le j' \le j$ and $i_1 \le i' < i_2$.

2.3 Lattice paths in signed permutations grids

As a preparation for counting the number of grid points of certain d-types, we connect grid points with the same d_h-type p or d_v-type q for $p, q \in \{0, 1\}$ by specific rules to build p_h-paths or q_v-paths.

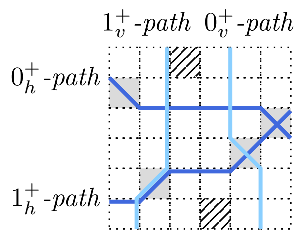
Definition 2.10. *Let $\pi \in \mathfrak{B}_n$ and P_π be its grid, then for $p, q \in \{0, 1\}$,*

(a) the p_h^+-paths (reps. p_h^--paths) are established by connecting grid points of d_h^+-types (resp. d_h^--types) p only along horizontal grid lines, and then connecting these segments of length less than n by diagonal lines only if they touch the same filled square;

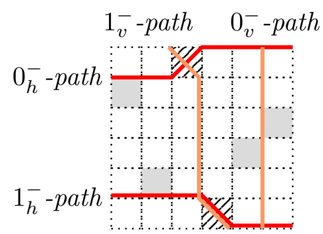
(b) the q_v^+-paths (reps. q_v^--paths) are established by connecting grid points of d_v^+-types (resp. d_v^--types) q only along vertical grid lines, and then connecting these segments of length less than n by diagonal lines only if they touch the same filled square.

We refer p_h^+-paths and p_h^--paths collectively as p_h-paths, and refer q_v^+-paths and q_v^--paths collectively as q_v-paths. By the construction above, we know that for any grid point, it must be on exactly one p_h^+-path, one p_h^--path, one q_v^+-path and one q_v^--path.

Example 2.11. *The following figures present each one of p_x^*-path in the permutation grid of $\pi = \bar{3}1652\bar{4}$, where $p \in \{0, 1\}$, $x \in \{h, v\}$ and $* \in \{+, -\}$.*



P_π



P_π

The above example also present the walking trends of these p_h-paths and the q_v-paths for $p, q \in \{0, 1\}$.

Theorem 2.12. *Let $\pi \in \mathfrak{B}_n$, then in the grid P_π,*

(a) each 0_h^+-path (resp. 1_h^+-path) goes from the left boundary to the right boundary along the horizontal grid lines except for carrying out a southeast (resp. northeast) step when encountering a positive square;

(b) each 0_v^+-path (resp. 1_v^+-path) goes from the top boundary to the bottom boundary along the vertical grid lines except for carrying out a southeast (resp. southwest) step when encountering a positive square;

(c) each 0_h^--path (resp. 1_h^--path) goes from the left boundary to the right boundary along the horizontal grid lines except for carrying out a northeast (resp. southeast) step when encountering a negative square;

(d) *each 0_v^--path (resp. 1_v^--path) goes from the top boundary to the bottom boundary along the vertical grid lines except for carrying out a southwest (resp. southeast) step when encountering a negative square.*

Proof. By Propositions 2.8 and 2.9, we note that the d_h-types (resp. d_v-types) on the same horizontal (resp. vertical) grid line change from 0 to 1 or from 1 to 0 only when it touches a filled square of the same sign. Thus in constructing processes in Definition 2.10, after connecting the grid points of the same d_h-type or d_v-type by horizontal or vertical grid lines, those d_h-segments or d_v-segments of length less than $n+1$ must touch filled squares with the same sign as the d_h-type or d_v-type.

Hence, for $p, q \in \{0,1\}$, connecting the p_h-segments or q_v-segments by diagonal lines in the filled square of the same sign as p_h or q_v is equivalent to that the p_h-path or the q_v-path taking a diagonal step when it meets that square, and the direction of this diagonal step is completely determined by Proposition 2.7. ∎

Based on the above observations, for $p, q \in \{0,1\}$, the numbers of $\overline{B}$-descents and $\overline{B}$-idescents can be utilized to count the numbers of p_h-paths and q_v-paths, respectively.

Theorem 2.13. *Let $\pi \in \mathfrak{B}_n$, then in the grid P_π,*

(a) *the number of 0_h^+-paths is* $\mathrm{des}^B(\pi)+1$,

(b) *the number of 0_h^--paths is* $\mathrm{des}^B(\pi)$,

(c) *the number of 1_h^+-paths is* $n-\mathrm{des}^B(\pi)$,

(d) *the number of 1_h^--paths is* $n-\mathrm{des}^B(\pi)+1$;

and

(e) *the number of 0_v^+-paths is* $\mathrm{ides}^B(\pi)+1$,

(f) *the number of 0_v^--paths is* $\mathrm{ides}^B(\pi)$,

(g) *the number of 1_v^+-paths is* $n-\mathrm{ides}^B(\pi)$,

(h) *the number of 1_v^--paths is* $n-\mathrm{ides}^B(\pi)+1$.

Proof. By Theorem 2.12, for $p, q \in \{0,1\}$, the number of p_h-paths equals the number of grid points $(i,1)$ of d_h-type p for $1 \le i \le n+1$, and the number of q_v-paths equals the number of grid points $(1,j)$ of d_v-type q for $1 \le j \le n+1$.

We first assume $\pi_i > 0$ for all $1 \le i \le n$. Hence in the grid P_π, there are only four types of pairs of filled squares: des_+^+-pair, asc_+^+-pair, ides_+^+-pair and iasc_+^+-pair, which are counted by $\mathrm{des}^B(\pi)$, $n-1-\mathrm{des}^B(\pi)$, $\mathrm{ides}^B(\pi)$ and $n-1-\mathrm{ides}^B(\pi)$, respectively.

It follows from Proposition 2.3 that for $2 \le i \le n$, the numbers of grid points $(i,1)$ with $d_h^+(i,1)=0$, $d_h^-(i,1)=0$, $d_h^+(i,1)=1$ and $d_h^-(i,1)=1$ are $\mathrm{des}^B(\pi)$, $\mathrm{des}^B(\pi)$, $n-1-\mathrm{des}^B(\pi)$ and $n-1-\mathrm{des}^B(\pi)$, respectively. Since $\pi_1 > 0$ and $\pi_n > 0$, by Proposition 2.5, we have $d_h^+(1,1)=0$, $d_h^-(1,1)=1$, $d_h^+(n+1,1)=1$ and $d_h^-(n+1,1)=1$. Therefore, for $1 \le i \le n+1$, the number of grid points $(i,1)$ with $d_h^+(i,1)=0$, $d_h^-(i,1)=0$, $d_h^+(i,1)=1$ and $d_h^-(i,1)=1$ are $\mathrm{des}^B(\pi)+1$, $\mathrm{des}^B(\pi)$, $n-\mathrm{des}^B(\pi)$ and $n+1-\mathrm{des}^B(\pi)$, respectively.

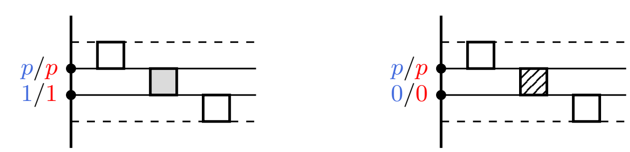
On the other side, by Proposition 2.4, for $2 \le j \le n$, the number of grid points $(1,j)$ with $d_v^+(1,j) = 0$, $d_v^-(1,j) = 0$, $d_v^+(1,j) = 1$ and $d_v^-(1,j) = 1$ are given by $\mathrm{ides}^B(\pi)$, $\mathrm{ides}^B(\pi)$, $n - 1 - \mathrm{ides}^B(\pi)$ and $n - 1 - \mathrm{ides}^B(\pi)$, respectively. Note that $d_v^+(1,1) = 0$, $d_v^-(1,1) = 1$, $d_v^+(1,n+1) = 1$ and $d_v^-(1,n+1) = 1$ by Proposition 2.5. Thus, for $1 \le j \le n+1$, the number of grid points $(1,j)$ with $d_v^+(1,j) = 0$, $d_v^-(1,j) = 0$, $d_v^+(1,j) = 1$ and $d_v^-(1,j) = 1$ are $\mathrm{ides}^B(\pi)+1$, $\mathrm{ides}^B(\pi)$, $n - \mathrm{ides}^B(\pi)$ and $n+1-\mathrm{ides}^B(\pi)$, respectively. So the theorem holds provided all filled squares in the grid P_π are positive.

To complete the proof, we are about to show that the quantitative relationships between p_h-paths and $\overline{B}$-descents, and between q_v-paths and $\overline{B}$-idescents, remain unchanged after any positive square is replaced by a negative square in P_π.

Let $\langle i,j\rangle$ with $1 \le i,j \le n$ be the replaced square and π' be the signed permutation after the replacement. In particular, when the positive square $\langle i,j\rangle$ is replaced, the effect is limited to the neighborhood formed by itself and the squares on its adjacent rows or columns, which corresponds to the 16 cases listed below. We only offer a detailed proof for Case 1, while the validity of the remaining cases can be also checked.

The symbols in the following proof are explained as: the filled squares in P_π and $P_{\pi'}$ indicate the square $\langle i,j\rangle$ before and after the substitution; the blank squares in P_π and $P_{\pi'}$ indicate the filled squares with undefined signs that remain unchanged after the substitution; the numbers a/b indicates the d_h^+-type (resp. d_v^+-type) a and d_h^--type (resp. d_v^--type) b of the corresponding grid point on the first column (resp. row); $\#p_x^*$ and $\#p_x^{*\,\prime}$ denote the numbers of p_x^*-paths in P_π and $P_{\pi'}$, respectively, where $p \in \{0,1\}$, $x \in \{h,v\}$ and $* \in \{+,-\}$.

Case 1: $|\pi_{i-1}| < \pi_i < |\pi_{i+1}|$ **for** $2 \le i \le n-1$. By Proposition 2.3, although the signs of π_{i-1} and π_{i+1} are not known, the squares $\langle i-1, |\pi_{i-1}|\rangle$ and $\langle i, \pi_i\rangle$ in P_π always changes from an asc-pair to a des-pair in $P_{\pi'}$ but preserve the d_h^+-type and d_h^--type of the grid point $(i,1)$, and the descent type of the pair $\langle i,\pi_i\rangle$ and $\langle i+1, |\pi_{i+1}|\rangle$ remains the same but the d_h^+-type and d_h^--type of $(i+1,1)$ both change from 1 to 0.



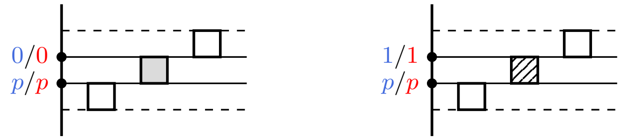
P_π $\quad\quad$ $P_{\pi'}$

square pairs	$\pi_{i-1} > 0/\pi_{i+1} > 0$	$\pi_{i-1} < 0/\pi_{i+1} < 0$
$\langle i-1, \lvert\pi_{i-1}\rvert\rangle$, $\langle i,\pi_i\rangle$	asc_+^+-pair $\to$ des_-^+-pair	asc_+^--pair $\to$ des_-^--pair
$\langle i,\pi_i\rangle$, $\langle i+1, \lvert\pi_{i+1}\rvert\rangle$	asc_+^+-pair $\to$ asc_+^--pair	des_-^+-pair $\to$ des_-^--pair

Thus, we obtain the desired change in the quantitative relationships, which are $\mathrm{des}^B(\pi') = \mathrm{des}^B(\pi) + 1$, and

$$\#0_h^{+\,\prime} = \#0_h^+ + 1,\ \#0_h^{-\,\prime} = \#0_h^- + 1,\ \#1_h^{+\,\prime} = \#1_h^+ - 1,\ \#1_h^{-\,\prime} = \#1_h^- - 1.$$

Case 2: $|\pi_{i+1}| < \pi_i < |\pi_{i-1}|$ **for** $2 \le i \le n-1$.



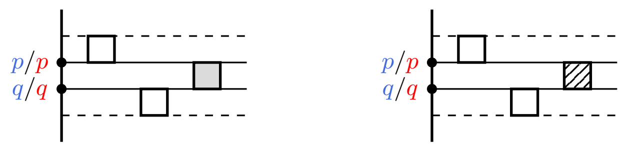
P_π $P_{\pi'}$

square pairs	$\pi_{i-1} > 0/\pi_{i+1} > 0$	$\pi_{i-1} < 0/\pi_{i+1} < 0$
$\langle i-1, \|\pi_{i-1}\|\rangle$, $\langle i, \pi_i\rangle$	des$^+_+$-pair $\to$ des$^-_+$-pair	asc$^+_-$-pair $\to$ asc$^-_-$-pair
$\langle i, \pi_i\rangle$, $\langle i+1, \|\pi_{i+1}\|\rangle$	des$^+_+$-pair $\to$ asc$^+_-$-pair	des$^-_+$-pair $\to$ asc$^-_-$-pair

After replacing the positive square $\langle i, \pi_i\rangle$ by a negative square, we have $\mathrm{des}^B(\pi') = \mathrm{des}^B(\pi) - 1$, and

$$\#0_h^{+\prime} = \#0_h^+ - 1,\ \#0_h^{-\prime} = \#0_h^- - 1,\ \#1_h^{+\prime} = \#1_h^+ + 1,\ \#1_h^{-\prime} = \#1_h^- + 1.$$

Case 3: $|\pi_{i-1}| < \pi_i$ **and** $|\pi_{i+1}| < \pi_i$ **for** $2 \le i \le n-1$.



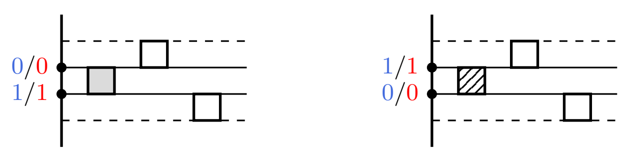
P_π $P_{\pi'}$

square pairs	$\pi_{i-1} > 0/\pi_{i+1} > 0$	$\pi_{i-1} < 0/\pi_{i+1} < 0$
$\langle i-1, \|\pi_{i-1}\|\rangle$, $\langle i, \pi_i\rangle$	asc$^+_+$-pair $\to$ des$^+_-$-pair	asc$^-_+$-pair $\to$ des$^-_-$-pair
$\langle i, \pi_i\rangle$, $\langle i+1, \|\pi_{i+1}\|\rangle$	des$^+_+$-pair $\to$ asc$^+_-$-pair	des$^-_+$-pair $\to$ asc$^-_-$-pair

After replacing the positive square $\langle i, \pi_i\rangle$ by a negative square, we have $\mathrm{des}^B(\pi') = \mathrm{des}^B(\pi)$, and

$$\#0_h^{+\prime} = \#0_h^+,\ \#0_h^{-\prime} = \#0_h^-,\ \#1_h^{+\prime} = \#1_h^+,\ \#1_h^{-\prime} = \#1_h^-.$$

Case 4: $\pi_i < |\pi_{i-1}|$ **and** $\pi_i < |\pi_{i+1}|$ **for** $2 \le i \le n-1$.



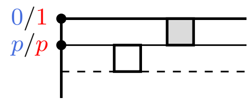
P_π $P_{\pi'}$

square pairs	$\pi_{i-1} > 0/\pi_{i+1} > 0$	$\pi_{i-1} < 0/\pi_{i+1} < 0$
$\langle i-1, \|\pi_{i-1}\|\rangle$, $\langle i, \pi_i\rangle$	des$^+_+$-pair $\to$ des$^-_+$-pair	asc$^+_-$-pair $\to$ asc$^-_-$-pair
$\langle i, \pi_i\rangle$, $\langle i+1, \|\pi_{i+1}\|\rangle$	asc$^+_+$-pair $\to$ asc$^-_+$-pair	des$^+_-$-pair $\to$ des$^-_-$-pair

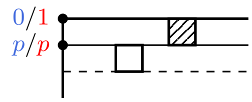
After replacing the positive square $\langle i, \pi_i \rangle$ by a negative square, we have $\mathrm{des}^B(\pi') = \mathrm{des}^B(\pi)$, and

$$\#0_h^{+\prime} = \#0_h^+,\ \#0_h^{-\prime} = \#0_h^-,\ \#1_h^{+\prime} = \#1_h^+,\ \#1_h^{-\prime} = \#1_h^-.$$

Case 5: $|\pi_{i+1}| < \pi_i$ for $i = 1$.



P_π



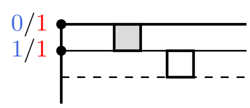
$P_{\pi'}$

square (pair)	$\pi_2 > 0$	$\pi_2 < 0$
$\langle 1, \pi_1 \rangle$	asc^+-square $\to$ des^--square	
$\langle 1, \pi_1 \rangle$, $\langle 2, \lvert\pi_2\rvert \rangle$	des_+^+-pair $\to$ asc_-^+-pair	des_+^--pair $\to$ asc_-^--pair

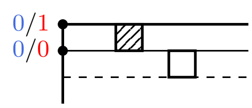
After replacing the positive square $\langle i, \pi_i \rangle$ by a negative square, we have $\mathrm{des}^B(\pi') = \mathrm{des}^B(\pi)$, and

$$\#0_h^{+\prime} = \#0_h^+,\ \#0_h^{-\prime} = \#0_h^-,\ \#1_h^{+\prime} = \#1_h^+,\ \#1_h^{-\prime} = \#1_h^-.$$

Case 6: $|\pi_i| < \pi_{i+1}$ for $i = 1$.



P_π



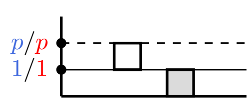
$P_{\pi'}$

square (pair)	$\pi_2 > 0$	$\pi_2 < 0$
$\langle 1, \pi_1 \rangle$	asc^+-square $\to$ des^--square	
$\langle 1, \pi_1 \rangle$, $\langle 2, \lvert\pi_2\rvert \rangle$	asc_+^+-pair $\to$ asc_+^--pair	des_-^+-pair $\to$ des_-^--pair

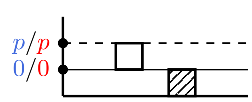
After replacing the positive square $\langle i, \pi_i \rangle$ by a negative square, we have $\mathrm{des}^B(\pi') = \mathrm{des}^B(\pi) + 1$, and

$$\#0_h^{+\prime} = \#0_h^+ + 1,\ \#0_h^{-\prime} = \#0_h^- + 1,\ \#1_h^{+\prime} = \#1_h^+ - 1,\ \#1_h^{-\prime} = \#1_h^- - 1.$$

Case 7: $|\pi_{i-1}| < \pi_i$ for $i = n$.



P_π



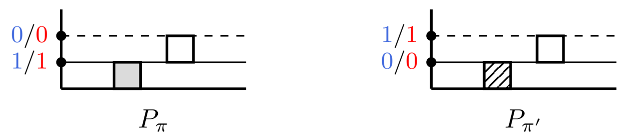
$P_{\pi'}$

square pair	$\pi_{n-1} > 0$	$\pi_{n-1} < 0$
$\langle n-1, \lvert\pi_{n-1}\rvert \rangle$, $\langle n, \pi_n \rangle$	asc_+^+-pair $\to$ des_-^+-pair	asc_+^--pair $\to$ des_-^--pair

After replacing the positive square $\langle i, \pi_i \rangle$ by a negative square, we have $\mathrm{des}^B(\pi') = \mathrm{des}^B(\pi) + 1$, and

$$\#0_h^{+\prime} = \#0_h^{+} + 1,\ \#0_h^{-\prime} = \#0_h^{-} + 1,\ \#1_h^{+\prime} = \#1_h^{+} - 1,\ \#1_h^{-\prime} = \#1_h^{-} - 1.$$

Case 8: $\boldsymbol{\pi_i < |\pi_{i-1}|}$ for $\boldsymbol{i = n}$.

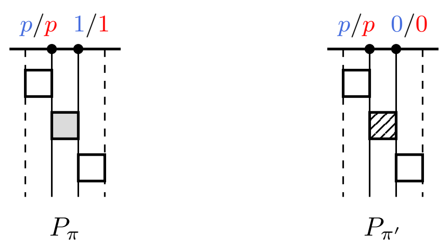


square pair	$\pi_{n-1} > 0$	$\pi_{n-1} < 0$
$\langle n-1, \|\pi_{n-1}\| \rangle$, $\langle n, \pi_n \rangle$	des_+^+-pair $\to$ des_+^--pair	asc_-^+-pair $\to$ asc_-^--pair

After replacing the positive square $\langle i, \pi_i \rangle$ by a negative square, we have $\mathrm{des}^B(\pi') = \mathrm{des}^B(\pi)$, and

$$\#0_h^{+\prime} = \#0_h^{+},\ \#0_h^{-\prime} = \#0_h^{-},\ \#1_h^{+\prime} = \#1_h^{+},\ \#1_h^{-\prime} = \#1_h^{-}.$$

Case 9: $\boldsymbol{|\pi_{j-1}^{-1}| < \pi_j^{-1} < |\pi_{j+1}^{-1}|}$ for $\boldsymbol{2 \le j \le n-1}$.

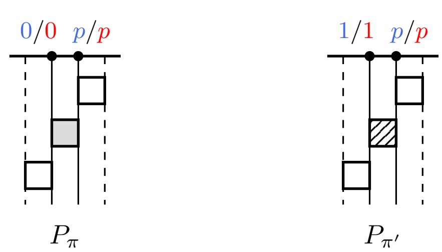


square pairs	$\pi_{j-1}^{-1} > 0/\pi_{j+1}^{-1} > 0$	$\pi_{j-1}^{-1} < 0/\pi_{j+1}^{-1} < 0$
$\langle \|\pi_{j-1}^{-1}\|, j-1 \rangle$, $\langle \pi_j^{-1}, j \rangle$	iasc_+^+-pair $\to$ ides_-^+-pair	iasc_+^--pair $\to$ ides_-^--pair
$\langle \pi_j^{-1}, j \rangle$, $\langle \|\pi_{j+1}^{-1}\|, j+1 \rangle$	iasc_+^+-pair $\to$ iasc_+^--pair	ides_-^+-pair $\to$ ides_-^--pair

After replacing the positive square $\langle \pi_j^{-1}, j \rangle$ by a negative square, we have $\mathrm{ides}^B(\pi') = \mathrm{ides}^B(\pi) + 1$, and

$$\#0_v^{+\prime} = \#0_v^{+} + 1,\ \#0_v^{-\prime} = \#0_v^{-} + 1,\ \#1_v^{+\prime} = \#1_v^{+} - 1,\ \#1_v^{-\prime} = \#1_v^{-} - 1.$$

Case 10: $\boldsymbol{|\pi_{j+1}^{-1}| < \pi_j^{-1} < |\pi_{j-1}^{-1}|}$ for $\boldsymbol{2 \le j \le n-1}$.

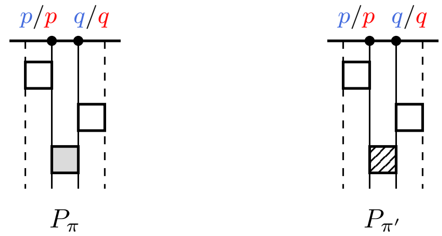


square pairs	$\pi_{j-1}^{-1}>0/\pi_{j+1}^{-1}>0$	$\pi_{j-1}^{-1}<0/\pi_{j+1}^{-1}<0$
$\langle\lvert\pi_{j-1}^{-1}\rvert, j-1\rangle$, $\langle\pi_j^{-1}, j\rangle$	ides_+^+-pair $\to$ ides_+^--pair	iasc_-^+-pair $\to$ iasc_-^--pair
$\langle\pi_j^{-1}, j\rangle$, $\langle\lvert\pi_{j+1}^{-1}\rvert, j+1\rangle$	ides_+^+-pair $\to$ iasc_-^+-pair	ides_+^--pair $\to$ iasc_-^--pair

After replacing the positive square $\langle\pi_j^{-1}, j\rangle$ by a negative square, we have $\mathrm{ides}^B(\pi') = \mathrm{ides}^B(\pi) - 1$, and

$$\#0_v^{+\prime} = \#0_v^{+} - 1,\ \#0_v^{-\prime} = \#0_v^{-} - 1,\ \#1_v^{+\prime} = \#1_v^{+} + 1,\ \#1_v^{-\prime} = \#1_v^{-} + 1.$$

Case 11: $|\pi_{j+1}^{-1}| < \pi_j^{-1}$ and $|\pi_{j-1}^{-1}| < \pi_j^{-1}$ for $2 \le j \le n-1$.

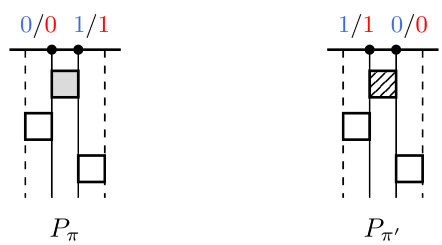


square pairs	$\pi_{j-1}^{-1}>0/\pi_{j+1}^{-1}>0$	$\pi_{j-1}^{-1}<0/\pi_{j+1}^{-1}<0$
$\langle\lvert\pi_{j-1}^{-1}\rvert, j-1\rangle$, $\langle\pi_j^{-1}, j\rangle$	iasc_+^+-pair $\to$ ides_-^+-pair	iasc_+^--pair $\to$ ides_-^--pair
$\langle\pi_j^{-1}, j\rangle$, $\langle\lvert\pi_{j+1}^{-1}\rvert, j+1\rangle$	ides_+^+-pair $\to$ iasc_-^+-pair	ides_+^--pair $\to$ iasc_-^--pair

After replacing the positive square $\langle\pi_j^{-1}, j\rangle$ by a negative square, we have $\mathrm{ides}^B(\pi') = \mathrm{ides}^B(\pi)$, and

$$\#0_v^{+\prime} = \#0_v^{+},\ \#0_v^{-\prime} = \#0_v^{-},\ \#1_v^{+\prime} = \#1_v^{+},\ \#1_v^{-\prime} = \#1_v^{-}.$$

Case 12: $\pi_j^{-1} < |\pi_{j+1}^{-1}|$ and $\pi_j^{-1} < |\pi_{j-1}^{-1}|$ for $2 \le j \le n-1$.

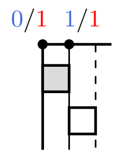


square pairs	$\pi_{j-1}^{-1}>0/\pi_{j+1}^{-1}>0$	$\pi_{j-1}^{-1}<0/\pi_{j+1}^{-1}<0$
$\langle\lvert\pi_{j-1}^{-1}\rvert, j-1\rangle$, $\langle\pi_j^{-1}, j\rangle$	ides_+^+-pair $\to$ ides_+^--pair	iasc_-^+-pair $\to$ iasc_-^--pair
$\langle\pi_j^{-1}, j\rangle$, $\langle\lvert\pi_{j+1}^{-1}\rvert, j+1\rangle$	iasc_+^+-pair $\to$ iasc_+^--pair	ides_-^+-pair $\to$ ides_-^--pair

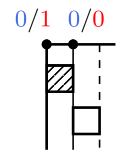
After replacing the positive square $\langle \pi_j^{-1}, j\rangle$ by a negative square, we have $\mathrm{ides}^B(\pi') = \mathrm{ides}^B(\pi)$, and

$$\#0_v^{+\prime} = \#0_v^+,\ \#0_v^{-\prime} = \#0_v^-,\ \#1_v^{+\prime} = \#1_v^+,\ \#1_v^{-\prime} = \#1_v^-.$$

Case 13: $\pi_j^{-1} < |\pi_{j+1}^{-1}|$ for $j = 1$.



P_π



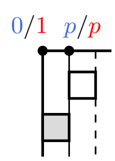
$P_{\pi'}$

square (pair)	$\pi_2^{-1} > 0$	$\pi_2^{-1} < 0$
$\langle \pi_1^{-1}, 1\rangle$	iasc^+-square $\to$ ides^--square	
$\langle \pi_1^{-1}, 1\rangle$, $\langle \lvert\pi_2^{-1}\rvert, 2\rangle$	iasc_+^+-pair $\to$ iasc_+^--pair	ides_-^+-pair $\to$ ides_-^--pair

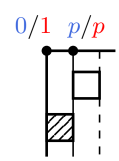
After replacing the positive square $\langle \pi_j^{-1}, j\rangle$ by a negative square, we have $\mathrm{ides}^B(\pi') = \mathrm{ides}^B(\pi) + 1$, and

$$\#0_v^{+\prime} = \#0_v^+ + 1,\ \#0_v^{-\prime} = \#0_v^- + 1,\ \#1_v^{+\prime} = \#1_v^+ - 1,\ \#1_v^{-\prime} = \#1_v^- - 1.$$

Case 14: $|\pi_{j+1}^{-1}| < \pi_j^{-1}$ for $j = 1$.



P_π



$P_{\pi'}$

square (pair)	$\pi_2^{-1} > 0$	$\pi_2^{-1} < 0$
$\langle \pi_1^{-1}, 1\rangle$	iasc^+-square $\to$ ides^--square	
$\langle \pi_1^{-1}, 1\rangle$, $\langle \lvert\pi_2^{-1}\rvert, 2\rangle$	ides_+^+-pair $\to$ iasc_-^+-pair	ides_+^--pair $\to$ iasc_-^--pair

After replacing the positive square $\langle \pi_j^{-1}, j\rangle$ by a negative square, we have $\mathrm{ides}^B(\pi') = \mathrm{ides}^B(\pi)$, and

$$\#0_v^{+\prime} = \#0_v^+,\ \#0_v^{-\prime} = \#0_v^-,\ \#1_v^{+\prime} = \#1_v^+,\ \#1_v^{-\prime} = \#1_v^-.$$

Case 15: $\pi_j^{-1} < |\pi_{j-1}^{-1}|$ for $j = n$.

square pair	$\pi_{n-1}^{-1} > 0$	$\pi_{n-1}^{-1} < 0$
$\langle \lvert\pi_{n-1}^{-1}\rvert, n-1\rangle$, $\langle \pi_n^{-1}, n\rangle$	ides_+^+-pair $\to$ ides_+^--pair	iasc_-^+-pair $\to$ iasc_-^--pair

After replacing the positive square $\langle \pi_j^{-1}, j\rangle$ by a negative square, we have $\mathrm{ides}^B(\pi') = \mathrm{ides}^B(\pi)$, and

$$\#0_v^{+\prime} = \#0_v^+,\ \#0_v^{-\prime} = \#0_v^-,\ \#1_v^{+\prime} = \#1_v^+,\ \#1_v^{-\prime} = \#1_v^-.$$

Case 16: $|\pi_{j-1}^{-1}| < \pi_j^{-1}$ for $j = n$.

square pair	$\pi_{n-1}^{-1} > 0$	$\pi_{n-1}^{-1} < 0$
$\langle \lvert\pi_{n-1}^{-1}\rvert, n-1\rangle$, $\langle \pi_n^{-1}, n\rangle$	iasc_+^+-pair $\to$ ides_-^+-pair	iasc_+^--pair $\to$ ides_-^--pair

After replacing the positive square $\langle \pi_j^{-1}, j\rangle$ by a negative square, we have $\mathrm{ides}^B(\pi') = \mathrm{ides}^B(\pi) + 1$, and

$$\#0_v^{+\prime} = \#0_v^+ + 1,\ \#0_v^{-\prime} = \#0_v^- + 1,\ \#1_v^{+\prime} = \#1_v^+ - 1,\ \#1_v^{-\prime} = \#1_v^- - 1.$$

Therefore, by induction, we complete the proof. ∎

2.4 Enumerations of grid points of certain d-types

For $\pi \in \mathfrak{B}_n$, let $n(\pi)$ be the number of negative elements in $\pi_1\pi_2\cdots\pi_n$. With the help of Theorems 2.12 and 2.13, we obtain the number of grid points of given d^+-type or d^--type in the grid P_π by $\mathrm{des}^B(\pi)$, $\mathrm{ides}^B(\pi)$ and $n(\pi)$.

Theorem 2.14. *Let $\pi \in \mathfrak{B}_n$, then in the grid P_π, there are*

(a) $((\mathrm{des}^B(\pi)+1)(\mathrm{ides}^B(\pi)+1) - n(\pi) + n)$ *grid points of d^+-type*$(0,0)$,

(b) $((\mathrm{ides}^B(\pi)+1)(n - \mathrm{des}^B(\pi)) + n(\pi) - n)$ *grid points of d^+-type* $(1,0)$,

(c) $((\mathrm{des}^B(\pi)+1)(n - \mathrm{ides}^B(\pi)) + n(\pi) - n)$ *grid points of d^+-type* $(0,1)$,

(d) $((n-\mathrm{des}^B(\pi))(n-\mathrm{ides}^B(\pi))-n(\pi)+n)$ *grid points of* d^+*-type* $(1,1)$*;*

and

(e) $(\mathrm{des}^B(\pi)\,\mathrm{ides}^B(\pi)+n(\pi))$ *grid points of* d^-*-type*$(0,0)$*,*

(f) $(\mathrm{ides}^B(\pi)(n-\mathrm{des}^B(\pi)+1)-n(\pi))$ *grid points of* d^-*-type* $(1,0)$*,*

(g) $(\mathrm{des}^B(\pi)(n-\mathrm{ides}^B(\pi)+1)-n(\pi))$ *grid points of* d^-*-type* $(0,1)$*,*

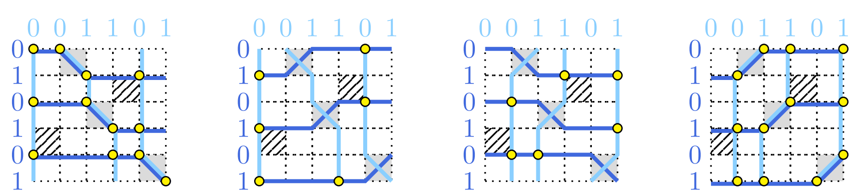
(h) $((n-\mathrm{des}^B(\pi)+1)(n-\mathrm{ides}^B(\pi)+1)+n(\pi))$ *grid points of* d^-*-type* $(1,1)$*.*

Proof. By Definition 2.10, for $p,q\in\{0,1\}$, a grid point is of d^+-type (resp. d^--type) (p,q) if and only if it is an intersection of a p_h^+-path (resp. p_h^--path) and a q_v^+-path (resp. q_v^--path). Note that the numbers of positive and negative squares in the grid P_π are $(n-n(\pi))$ and $n(\pi)$, respectively. Referring to Example 2.15, we will have a clearer picture of the proof that follows.

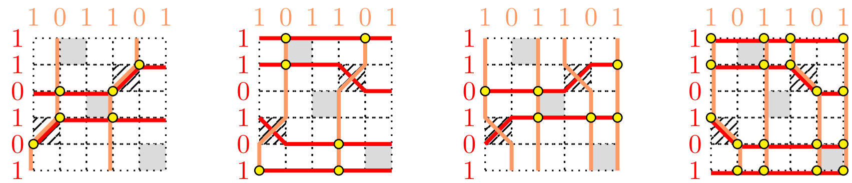
Since there are $(\mathrm{des}^B(\pi)+1)$ 0_h^+-paths and $(\mathrm{ides}^B(\pi)+1)$ 0_v^+-paths in P_π by Theorem 2.13, the 0_h^+-paths and 0_v^+-paths meet each other $(\mathrm{des}^B(\pi)+1)(\mathrm{ides}^B(\pi)+1)$ times. By noticing that both 0_h^+-paths and 0_v^+-paths take a southeast step when they encounter a positive square by Theorem 2.12, we see that the number of intersections formed by 0_h^+-paths and 0_v^+-paths at grid points is actually $(n-n(\pi))$, the number of positive squares, more than the number of times they meet. Thus we prove the statement (a). In terms of the counting formulas in Theorem 2.13, and similar analyses but replacing 0_h^+-paths with 1_h^+-paths, 0_h^--paths and 1_h^--paths respectively, and replacing 0_v^+-paths with 1_v^+-paths, 0_v^--paths and 1_v^--paths correspondingly, we can prove statements (d), (e) and (h).

To see the number of grid points of d^+-type $(1,0)$, we notice that if a 1_h^+-path and a 0_v^+-path meet in a positive square, their intersection is in the interior of the square instead of on the grid point, because by Theorem 2.12, the 1_h^+-path and the 0_v^+-path take the northeast and southeast steps, respectively, inside the square. Thus, the number of intersections on grid points formed by 1_h^+-paths and 0_v^+-paths is $(n-n(\pi))$ less than the number of times they meet. By Theorem 2.13, such intersections are counted by $((\mathrm{ides}^B(\pi)+1)(n-\mathrm{des}^B(\pi))+n(\pi)-n)$, which verifies statement (b). By replacing the 1_h^+-paths in the above proof process with 0_h^+-paths, 1_h^--paths, 0_h^--paths, and the 0_v^+-paths with 1_v^+-paths, 0_v^--paths, 1_v^--paths, respectively, and using the related quantities in Theorem 2.13, we can prove statements (c), (f), (g), respectively. ∎

Example 2.15. *Let* $\pi=2\bar{4}3\bar{1}5\in\mathfrak{B}_5$ *with* $\mathrm{des}^B(\pi)=2$*,* $\mathrm{ides}^B(\pi)=2$ *and* $n(\pi)=2$*, the following diagrams present the* d^+*-type and* d^-*-type of each grid point in the grid* P_π *by Theorem 2.14, where* $\#d^*_{(p,q)}$ *denote the number of grid points of* d^**-type* (p,q) *for* $*\in\{+,-\}$ *and* $p,q\in\{0,1\}$*.*



$\#d^+_{(0,0)}=12$ $\#d^+_{(1,0)}=6$ $\#d^+_{(0,1)}=6$ $\#d^+_{(1,1)}=12$



$\#d^{-}_{(0,0)} = 6$ $\quad \#d^{-}_{(1,0)} = 6$ $\quad \#d^{-}_{(0,1)} = 6$ $\quad \#d^{-}_{(1,1)} = 18$

3 The recurrence of $\overline{B}$-descents and $\overline{B}$-idescents on $\mathfrak{B}_n$

To give a combinatorial proof for the four-term recurrence (6) of $\bar{b}_{n,i,j}$ given in Theorem 1.2, we first introduce some sets of ordered pairs. Let

$$\mathcal{C}_{n,i,j} = \{(\sigma, k) \mid \sigma \in \overline{\mathcal{B}}_{n,i,j} \text{ and } 1 \le k \le n\},$$

and recall that

$$\overline{\mathcal{B}}_{n,i,j} = \{\sigma \in \mathfrak{B}_n \mid \ \mathrm{des}^B(\sigma) = i \text{ and } \mathrm{ides}^B(\sigma) = j\}.$$

Thus we have $|\mathcal{C}_{n,i,j}| = n\bar{b}_{n,i,j}$, which is equal to the left side of (6). Let

$$\mathcal{D}^{(p,q)^*}_{n,i,j} = \big\{(\pi, (r,s)) \mid \ \pi \in \overline{\mathcal{B}}_{n,i,j} \text{ and } d^*(r,s) = (p,q) \text{ in } P_\pi\big\} \tag{8}$$

with $* \in \{+,-\}$ and $p, q \in \{0,1\}$. Clearly, for any given $1 \le i,j \le n$, the eight sets $\mathcal{D}^{(0,0)^+}_{n,i,j}$, $\mathcal{D}^{(0,0)^-}_{n,i,j}$, $\mathcal{D}^{(0,1)^+}_{n,i,j-1}$, $\mathcal{D}^{(0,1)^-}_{n,i,j-1}$, $\mathcal{D}^{(1,0)^+}_{n,i-1,j}$, $\mathcal{D}^{(1,0)^-}_{n,i-1,j}$, $\mathcal{D}^{(1,1)^+}_{n,i-1,j-1}$ and $\mathcal{D}^{(1,1)^-}_{n,i-1,j-1}$ are pairwise disjoint, and by Theorem 2.14, we see

$$\left|\mathcal{D}^{(0,0)^+}_{n-1,i,j} \uplus \mathcal{D}^{(0,0)^-}_{n-1,i,j}\right| = (n+i+j+2ij)\bar{b}_{n-1,i,j},$$

$$\left|\mathcal{D}^{(0,1)^+}_{n-1,i,j-1} \uplus \mathcal{D}^{(0,1)^-}_{n-1,i,j-1}\right| = ((2n-1)i - (2i+1)(j-1))\bar{b}_{n-1,i,j-1},$$

$$\left|\mathcal{D}^{(1,0)^+}_{n-1,i-1,j} \uplus \mathcal{D}^{(1,0)^-}_{n-1,i-1,j}\right| = ((2n-1)j - (2j+1)(i-1))\bar{b}_{n-1,i-1,j},$$

$$\left|\mathcal{D}^{(1,1)^+}_{n-1,i-1,j-1} \uplus \mathcal{D}^{(1,1)^-}_{n-1,i-1,j-1}\right| = ((2n^2-n) + 2(i-1)(j-1) + (1-2n)(i+j-2))\bar{b}_{n-1,i-1,j-1},$$

where $\biguplus$ indicates the union of pairwise disjoint sets. Hence the cardinality of the set $\mathcal{D}_{n-1,i,j}$ defined as

$$\mathcal{D}_{n-1,i,j} = \biguplus_{p,q\in\{0,1\}} \mathcal{D}^{(p,q)^+}_{n-1,i-p,j-q} \uplus \mathcal{D}^{(p,q)^-}_{n-1,i-p,j-q}$$

is equal to the right side of (6).

Combinatorial Proof of Theorem 1.2. We shall establish a bijection between $\mathcal{D}_{n-1,i,j}$ and $\mathcal{C}_{n,i,j}$ by employing the inserting operation $\varphi_{(i,j)}$ or $\overline{\varphi}_{(i,j)}$. In particular, define

$$\Psi((\pi,(r,s))) = (\sigma, r)$$

for any pair $(\pi,(r,s)) \in \mathcal{D}_{n-1,i,j}$, where

$$\sigma = \begin{cases} \varphi_{(r,s)}(\pi), & \text{if } (\pi,(r,s)) \in \mathcal{D}^{(p,q)^+}_{n-1,i-p,j-q}\ , \\ \overline{\varphi}_{(r,s)}(\pi), & \text{if } (\pi,(r,s)) \in \mathcal{D}^{(p,q)^-}_{n-1,i-p,j-q}\ , \end{cases} \tag{9}$$

for $p,q \in \{0,1\}$.

By the construction as given in (8), for

$$((\pi,(r,s))) \in \mathcal{D}^{(p,q)^+}_{n-1,i-p,j-q} \uplus \mathcal{D}^{(p,q)^-}_{n-1,i-p,j-q},$$

we have $\pi \in \overline{\mathcal{B}}_{n-1,i-p,j-q}$, and the grid point (r,s) is of d^+-type or d^--type (p,q), which implies

$$(\mathrm{des}(\varphi_{(r,s)}(\pi)), \mathrm{ides}(\varphi_{(r,s)}(\pi))) - (\mathrm{des}(\pi), \mathrm{ides}(\pi)) = (p,q),$$

or

$$(\mathrm{des}(\overline{\varphi}_{(r,s)}(\pi)), \mathrm{ides}(\overline{\varphi}_{(r,s)}(\pi))) - (\mathrm{des}(\pi), \mathrm{ides}(\pi)) = (p,q)$$

by Definitions 2.1 and 2.2 of d-types. Thus we obtain $\sigma \in \mathcal{B}_{n,i,j}$, and $(\sigma,r) \in \mathcal{C}_{n,i,j}$.

On the other hand, for any pair $(\sigma,r) \in \mathcal{C}_{n,i,j}$, notice that the signed permutation σ has i descents and j idescents. Then deleting the square $\langle r, |\sigma_r| \rangle$ in the grid P_σ by the operation $\varphi^{-1}_{\langle r,\sigma_r \rangle}$ if $\sigma_r > 0$ and $\overline{\varphi}^{-1}_{\langle r,-\sigma_r \rangle}$ if $\sigma_r < 0$, we obtain a signed permutation π that must belong the set $\mathfrak{B}_{n,i-p,j-p}$ for some $p,q \in \{0,1\}$ by Proposition 2.6. Moreover, the grid point $(r,|\sigma_r|)$ in the grid P_π must have d^+-type (p,q) if $\sigma_r > 0$ or d^--type (p,q) if $\sigma_r < 0$. Thus, we deduce

$$(\pi,(r,|\sigma_r|)) \in \mathcal{D}^{(p,q)^+}_{n-1,i-p,j-q} \uplus \mathcal{D}^{(p,q)^-}_{n-1,i-p,j-q}\ ,$$

and the inverse of Ψ is given as

$$\Psi^{-1}((\sigma,r)) = (\pi,(r,|\sigma_r|))$$

for any pair $(\sigma,r) \in \mathcal{C}_{n,i,j}$, where

$$\pi = \begin{cases} \varphi^{-1}_{\langle r,\sigma_r \rangle}(\sigma), & \text{if } \sigma_r > 0, \\ \overline{\varphi}^{-1}_{\langle r,-\sigma_r \rangle}(\sigma), & \text{if } \sigma_r < 0. \end{cases} \tag{10}$$

Therefore, with the combination of (9) and (10), we get the desired bijection

$$\Psi : \mathcal{D}_{n-1,i,j} \leftrightarrow \mathcal{C}_{n,i,j},$$

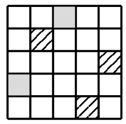
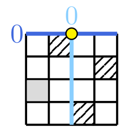
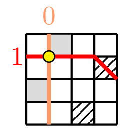
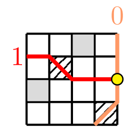
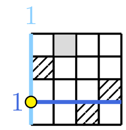
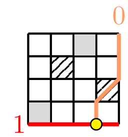
which completes the proof of Theorem 1.2. ∎

Example 3.1. *For $\sigma = 3\overline{25}1\overline{4} \in \overline{B}_{5,3,2}$, as shown below, for $1 \le r \le 5$, we have*

$$\Psi^{-1}(\sigma,1) = (\overline{24}1\overline{3},(1,3)) \in \mathcal{D}^{(0,0)^+}_{4,3,2}\ , \quad \Psi^{-1}(\sigma,2) = (2\overline{4}1\overline{3},(2,2)) \in \mathcal{D}^{(1,0)^-}_{4,2,2}\ ,$$

$$\Psi^{-1}(\sigma,3) = (3\overline{2}1\overline{4},(3,5)) \in \mathcal{D}^{(1,0)^-}_{4,2,2}\ , \quad \Psi^{-1}(\sigma,4) = (2\overline{143},(4,1)) \in \mathcal{D}^{(1,1)^+}_{4,2,1}\ ,$$

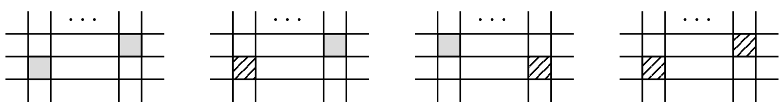
$$\Psi^{-1}(\sigma,5) = (2\overline{34}1,(5,4)) \in \mathcal{D}^{(1,0)^-}_{4,2,2}\ .$$

$\sigma \in \mathcal{B}_{5,3,2}$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
					
$3\bar{2}51\bar{4}$	$\bar{2}41\bar{3}$	$2\bar{4}1\bar{3}$	$3\bar{2}1\bar{4}$	$2\bar{1}4\bar{3}$	$2\bar{3}\bar{4}1$

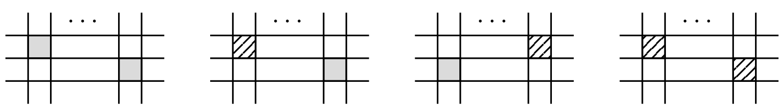
4 The recurrence of $\underline{B}$-descents and $\underline{B}$-idescents on $\mathfrak{B}_n$

In the subsequent contexts, the definitions and the terminologies is under the r-order, except for special mentions. Since techniques used in the proof are similar to the previous sections, we shall omit most of the detailed process and instead give the corresponding conclusion directly.

We still start by investigating the des-pairs, asc-pairs, and d-types. Comparing to the natural order (1), we note that the only difference is pair of the two negative squares. Thus, the following are the eight types of des-pairs and asc-pairs under the r-order.

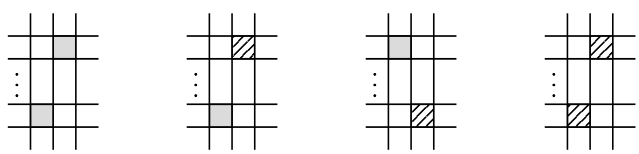


des$^+_+$-pair des$^-_+$-pair des$^+_-$-pair des$^-_-$-pair

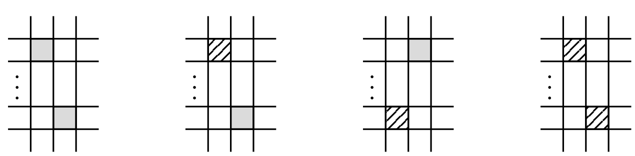


asc$^+_+$-pair asc$^-_+$-pair asc$^+_-$-pair asc$^-_-$-pair

And the following are the eight types of ides-pairs and iasc-pairs.



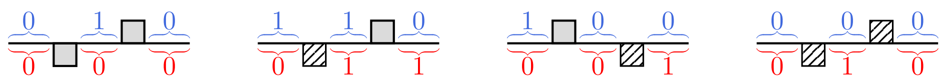
ides$^+_+$-pair ides$^-_+$-pair ides$^+_-$-pair ides$^-_-$-pair



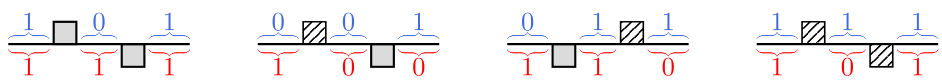
iasc$^+_+$-pair iasc$^-_+$-pair iasc$^+_-$-pair iasc$^-_-$-pair

By changing only des^B to des_B in Definitions 2.1 and 2.2, we get the definitions and related notations of $d_h^{\pm}$-types, $d_v^{\pm}$-types and $d^{\pm}$-types under the r-order. The following propositions describe the distributions of the $d_h^{\pm}$-types and $d_v^{\pm}$-types on the grid lines.

Proposition 4.1. *The d_h-types of the grid points on the middle horizontal line of des-pairs and asc-pairs in the grid are indicated in the figure below, where the numbers above are d_h^+-types, and the numbers below are d_h^--types.*

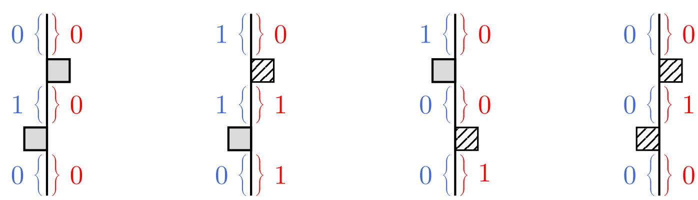


des_+^+-pair *des_+^--pair* *des_-^+-pair* *des_-^--pair*

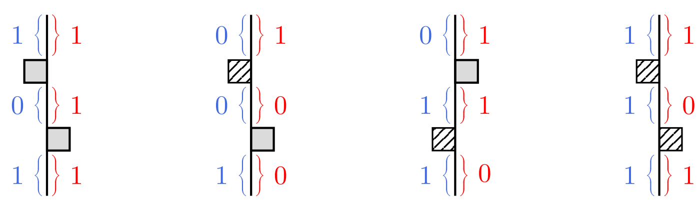


asc_+^+-pair *asc_+^--pair* *asc_-^+-pair* *asc_-^--pair*

Proposition 4.2. *The d_v-types of the grid points on the middle vertical line of ides-pairs and iasc-pairs in the grid are indicated in the figure below, where the numbers on the left are d_v^+-types, and the numbers on the right are d_v^--types.*



$ides_+^+$-pair *$ides_+^-$-pair* *$ides_-^+$-pair* *$ides_-^-$-pair*



$iasc_+^+$-pair *$iasc_+^-$-pair* *$iasc_-^+$-pair* *$iasc_-^-$-pair*

Proposition 4.3. *The d_h-types of the grid points on the top and bottom boundaries of the grid P_π are indicated in the figure below, where the numbers above are d_h^+-types, and the numbers below are d_h^--types.*

d_h-types on the top boundary *d_h-types on the bottom boundary*

And the d_v-types of the grid points on the left and right boundaries of the grid are indicated in the figure below, where the numbers on the left are d_v^+-types, and the numbers on the right are d_v^--types.

d_v-types on the left boundary *d_v-types on the right boundary*

Combining Propositions 4.1, 4.2 and 4.3, we see the d-types of the four corners of the filled squares in the permutation grid have fixed values.

Proposition 4.4. *Let $\pi \in \mathfrak{B}_n$ and P_π be its grid, then for $1 \le i, j \le n$, we have*

$$d^+(i,j) = d^+(i+1,j+1) = (0,0) \quad \textit{and} \quad d^+(i,j+1) = d^+(i+1,j) = (1,1),$$

if $\langle i,j \rangle$ is a positive square in P_π, and

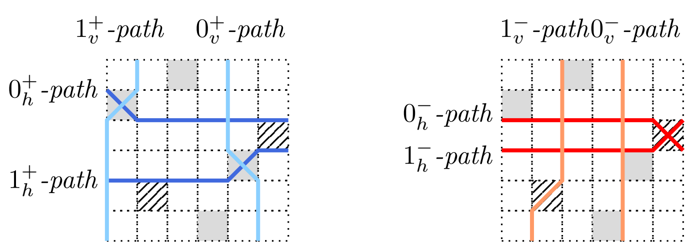
$$d^-(i,j) = d^-(i+1,j+1) = (0,0) \quad \textit{and} \quad d^-(i,j+1) = d^-(i+1,j) = (1,1).$$

and if $\langle i,j \rangle$ is a negative square in P_π.

d^+-types on a positive square *d^--types on a negative square*

Consistent with Definition 2.10, for $p, q \in \{0,1\}$, we can construct $p_h^\pm$-paths and $q_v^\pm$-paths based on $d_h^\pm$-types and $d_v^\pm$-types under the r-order.

Example 4.5. *The figures below present each one of p_x^*-paths in the permutation grid of $\pi = 31\bar{6}5\bar{2}4$, where $p \in \{0,1\}, x \in \{h,v\}$ and $* \in \{+,-\}$.*



P_π P_π

These paths defined by d-types under the r-order also exhibit similar behavior as those defined under the natural order when they encounter filled squares as described in Theorem 2.12.

Theorem 4.6. *Let $\pi \in \mathfrak{B}_n$, then in the signed permutation grid P_π,*

(a) each 0_h^+-path (resp. 1_h^+-path) goes from the left boundary to the right boundary along the horizontal grid lines except for carrying out a southeast (resp. northeast) step when encountering a positive square;

(b) each 0_v^+-path (resp. 1_v^+-path) goes from the top boundary to the bottom boundary along the vertical grid lines except for carrying out a southeast (resp. southwest) step when encountering a positive square;

(c) each 0_h^--path (resp. 1_h^--path) goes from the left boundary to the right boundary along the horizontal grid lines except for carrying out a southeast (resp. northeast) step when encountering a negative square;

(d) each 0_v^--path (resp. 1_v^--path) goes from the top boundary to the bottom boundary along the vertical grid lines except for carrying out a southeast (resp. southwest) step when encountering a negative square.

Following the proof of Theorem 2.13 but dealing with more complicated cases, we give the counting formulas for the paths in terms of $\mathrm{des}_B(\pi)$ and $\mathrm{ides}_B(\pi)$ with any given $\pi \in \mathfrak{B}_n$.

Theorem 4.7. *Let $\pi \in \mathfrak{B}_n$, then in the signed permutation grid P_π,*

(a) the number of 0_h^+-paths is $\mathrm{des}_B(\pi) + 1$,

(b) the number of 0_h^--paths is $\mathrm{des}_B(\pi)$,

(c) the number of 1_h^+-paths is $n - \mathrm{des}_B(\pi)$,

(d) the number of 1_h^--paths is $n - \mathrm{des}_B(\pi) + 1$;

and

(e) the number of 0_v^+-paths is $\mathrm{ides}_B(\pi) + 1$,

(f) the number of 0_v^--paths is $\mathrm{ides}_B(\pi)$,

(g) the number of 1_v^+-paths is $n - \mathrm{ides}_B(\pi)$,

(h) the number of 1_v^--paths is $n - \mathrm{ides}_B(\pi) + 1$.

By Theorems 4.6 and 4.7, in any grid P_π, we can use the statistics $\mathrm{des}_B(\pi)$, $\mathrm{ides}_B(\pi)$ and $n(\pi)$ to determine the number of grid points of given d^+-type or d^--type.

Theorem 4.8. *Let $\pi \in \mathfrak{B}_n$, then in the grid P_π, there are*

(a) $((\mathrm{des}_B(\pi) + 1)(\mathrm{ides}_B(\pi) + 1) - n(\pi) + n)$ grid points of d^+-type$(0, 0)$,

(b) $((\mathrm{ides}_B(\pi) + 1)(n - \mathrm{des}_B(\pi)) + n(\pi) - n)$ grid points of d^+-type $(1, 0)$,

(c) $((\mathrm{des}_B(\pi)+1)(n-\mathrm{ides}_B(\pi))+n(\pi)-n)$ *grid points of* d^+*-type* $(0,1)$,

(d) $((n-\mathrm{des}_B(\pi))(n-\mathrm{ides}_B(\pi))-n(\pi)+n)$ *grid points of* d^+*-type* $(1,1)$;

and

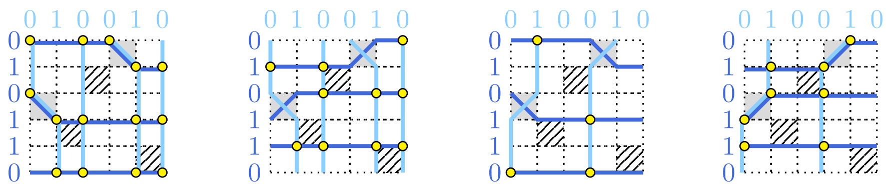
(e) $(\mathrm{des}_B(\pi)\,\mathrm{ides}_B(\pi)+n(\pi))$ *grid points of* d^-*-type*$(0,0)$,

(f) $(\mathrm{ides}_B(\pi)(n-\mathrm{des}_B(\pi)+1)-n(\pi))$ *grid points of* d^-*-type* $(1,0)$,

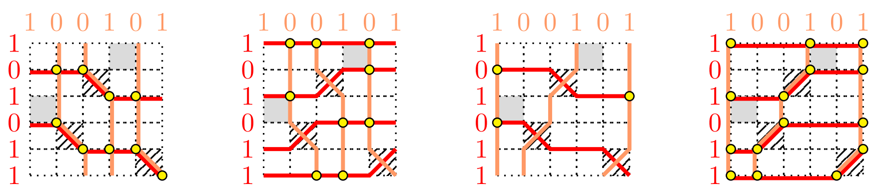
(g) $(\mathrm{des}_B(\pi)(n-\mathrm{ides}_B(\pi)+1)-n(\pi))$ *grid points of* d^-*-type* $(0,1)$,

(h) $((n-\mathrm{des}_B(\pi)+1)(n-\mathrm{ides}_B(\pi)+1)+n(\pi))$ *grid points of* d^-*-type* $(1,1)$.

Example 4.9. *Let* $\pi = 4\bar{3}1\bar{2}5 \in \mathfrak{B}_5$ *with* $des_B(\pi)=2$, $ides_B(\pi)=3$ *and* $n(\pi)=2$, *then the following diagrams show the* d^+*-type and* d^-*-type of each grid point, where* $\#d^*(p,q)$ *denote the number of grid points with* d^**-type* (p,q) *for* $*\in\{+,-\}$ *and* $p,q\in\{0,1\}$.



$\#d^+(0,0)=14$ $\#d^+(1,0)=10$ $\#d^+(0,1)=4$ $\#d^+(1,1)=8$



$\#d^-(0,0)=9$ $\#d^-(1,0)=9$ $\#d^-(0,1)=3$ $\#d^-(1,1)=15$

We now present the combinatorial proof of Theorem 1.4. Note that some symbols in the proof of Theorem 1.2 are reused , but keep in mind that the objects involved in these symbols are defined under the r-order.

Combinatorial Proof of Theorem 1.4. We have

$$\underline{\mathcal{B}}_{n,i,j} = \{\sigma\in\mathfrak{B}_n \mid \ \mathrm{des}_B(\sigma)=i \text{ and } \mathrm{ides}_B(\sigma)=j\}$$

and define

$$\mathcal{C}_{n,i,j} = \{(\sigma,k) \mid \sigma\in\underline{\mathcal{B}}_{n,i,j} \text{ and } 1\le k\le n\}$$

whose cardinality $|\mathcal{C}_{n,i,j}| = n\underline{b}_{n,i,j}$ is the left side of (7). For $*\in\{+,-\}$ and $p,q\in\{0,1\}$, let

$$\mathcal{D}_{n,i,j}^{(p,q)^*} = \left\{(\pi,(r,s)) \mid \ \pi\in\underline{\mathcal{B}}_{n,i,j} \text{ and } d^*(r,s)=(p,q) \text{ in } P_\pi\right\}.$$

Then it follows from Theorem 4.8 that the cardinality of the set

$$\mathcal{D}_{n-1,i,j} = \biguplus_{p,q\in\{0,1\}} \mathcal{D}_{n-1,i-p,j-q}^{(p,q)^+} \uplus \mathcal{D}_{n-1,i-p,j-q}^{(p,q)^-}$$

equals the right side of (7). Define

$$\Psi((\pi,(r,s))) = (\sigma, r),$$

where for $p,q \in \{0,1\}$,

$$\sigma = \begin{cases} \varphi_{(r,s)}(\pi), & \text{if } (\pi,(r,s)) \in \mathcal{D}^{(p,q)^+}_{n-1,i-p,j-q}\ , \\ \overline{\varphi}_{(r,s)}(\pi), & \text{if } (\pi,(r,s)) \in \mathcal{D}^{(p,q)^-}_{n-1,i-p,j-q}\ . \end{cases}$$

Clearly Ψ is a bijection between $\mathcal{D}_{n-1,i,j}$ and $\mathcal{C}_{n,i,j}$, which completes the proof. ∎

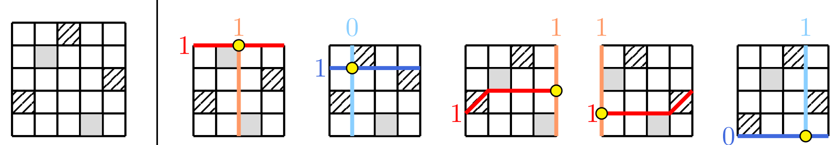
Example 4.10. *For* $\sigma = \bar{3}2\overline{51}4 \in \underline{B}_{5,3,3}$, *as shown below, for* $1 \le r \le 5$, *we have*

$$\Psi^{-1}(\sigma,1) = (2\overline{41}3,(1,3)) \in \mathcal{D}^{(1,1)^-}_{4,2,2}\ , \quad \Psi^{-1}(\sigma,2) = (\overline{241}3,(2,2)) \in \mathcal{D}^{(1,0)^+}_{4,2,3}\ ,$$

$$\Psi^{-1}(\sigma,3) = (\bar{3}2\bar{1}4,(3,5)) \in \mathcal{D}^{(1,1)^-}_{4,2,2}\ , \quad \Psi^{-1}(\sigma,4) = (\bar{2}1\bar{4}3,(4,1)) \in \mathcal{D}^{(1,1)^-}_{4,2,2}\ ,$$

$$\Psi^{-1}(\sigma,5) = (\bar{3}2\overline{41},(5,4)) \in \mathcal{D}^{(0,1)^+}_{4,3,2}\ .$$

$\sigma \in \mathcal{B}_{5,3,3}$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$\bar{3}2\overline{51}4$	$2\overline{41}3$	$\overline{241}3$	$\bar{3}2\bar{1}4$	$\bar{2}1\bar{4}3$	$\bar{3}2\overline{41}$



5 Further remark

Since the symmetric group $\mathfrak{S}_n$ is a subgroup of $\mathfrak{B}_n$, actually, our combinatorial proofs in Theorems 1.2 and 1.4 also induce a intuitive way to understand the following recurrence formula related to the joint distribution descents and inverse descents on $\mathfrak{S}_n$ (see [13, Eq.(10)]),

$$\begin{aligned} nA_{n,i,j} =& (ij+n-1)A_{n-1,i,j} + (1-n+j(n+1-i))A_{n-1,i-1,j} \\ &+ (1-n+i(n+1-j))A_{n-1,i,j-1} \\ &+ (n-1+(n+1-i)(n+1-j))A_{n-1,i-1,j-1}, \end{aligned} \tag{11}$$

where $A_{n,i,j}$ is the number of the permutations in $\mathfrak{S}_n$ with $(i-1)$ descents and $(j-1)$ idescents for $1 \le i,j \le n$. The formula was first obtained by Carlitz, Roselle, and Scoville [5] when they studied the generating function of $A_{n,i,j}$,

$$A_n(s,t) = \sum_{w \in \mathfrak{S}_n} s^{\mathrm{des}(w)+1} t^{\mathrm{ides}(w)+1} = \sum_{i,j=1}^{n} A_{n,i,j} s^i t^j.$$

By the model of balls in boxes, Petersen [13] gave an expansion formula of $\frac{A_{s,t}}{(1-s)^{n+1}(1-t)^{n+1}}$ and used it to reprove (11). We mention that their methods are algebraic or semicombinatorial, and a pure combinatorial proof for (11) is presented in [9].

With the help of quasisymmetric functions, Moustakas [10] derived a recurrence formula of $\underline{B}$-descents over involutions in $\mathfrak{B}_n$, and Cao-Liu [6] obtained a recurrence formula of $\underline{B}$-descents over fixed-point free involutions in $\mathfrak{B}_n$. Since the descents are coincidence with the idescents in involutions, in [8], by utilizing the enumerative results of lattice paths in signed permutations grids, we not only give bijective proofs of these two formulas of Moustakas and Cao-Liu, but also determine the recursive formulas of $\overline{B}$-descents over involutions and fixed-point free involutions in $\mathfrak{B}_n$, respectively.

Notice that by reversing only all the negative elements in signed permutations, we can directly see that des^B and des_B are equally distributed over $\mathfrak{B}_n$. Thus we are curious how to find a straightforward bijection to prove the equidistribution property of $(\mathrm{des}^B, \mathrm{ides}^B)$ and $(\mathrm{des}_B, \mathrm{ides}_B)$, or to get further refinements of the equidistribution property by involving other statistics such as major indices, descent sets, and so on.

Acknowledgement *We thank the reviewer for his/her comments, which have greatly improved the quality of this manuscript. The second author is supported by the National Natural Science Foundation of China (12001078), the Natural Science Foundation of Chongqing (CSTB2022NSCQ-MSX0465), and the Doctoral Research Start-up Funding of CQUPT (A2020-31). The last author is supported by the National Natural Science Foundation of China (11901074), and the Natural Science Foundation of Chongqing (CSTB2023NSCQ-MSX0666).*

References

[1] R. M. Adin, C. A. Athanasiadis, S. Elizalde, Y. Roichman, Character formulas and descents for the hyperoctahedral group, *Adv. Appl. Math.* **87** (2017), 128–169.

[2] R. M. Adin, F. Brenti, Y. Roichman, Descent numbers and major indices for the hyperoctahedral group, *Adv. Appl. Math.* **27** (2001), 210–224.

[3] C. A. Athanasiadis, Gamma-positivity in combinatorics and geometry, *Sémin. Lothar. Comb.* **77** (2018), B77i.

[4] A. Björner, F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics **231**, Springer, New York (2005).

[5] L. Carlitz, D. P. Roselle, R. A. Scoville, Permutations and sequences with repetitions by number of increases, *J. Combin. Theory* **1** (1966), 350–374.

[6] J. Cao, L. L. Liu, The Eulerian distribution on the fixed-point free involutions of the hyperoctahedral group, *J. Algebraic Combin.* **57** (2023), 793–810.

[7] C.-O. Chow, On certain combinatorial expansions of the Eulerian polynomials, *Adv. Appl. Math.* **41** (2008), 133–157.

[8] X. Gao, F. Z. K. LI, L. Wan, J. Y. X. Yang, On the combinatorics of descents and inverse descents over involutions in the hyperoctahedral group, in preparation.

[9] F. Z. K. Li, X. Liu, Combinatorial proofs for the joint distribution of descents and inverse descents, to appear in *Rocky Mountain J. Math.*

[10] V. D. Moustakas, The Eulerian distribution on the involutions of the hyperoctahedral group is unimodal, *Graphs Combin.* **35** (2019), 1077-1090.

[11] V. D. Moustakas, *Enumerative Combinatorics, Representations and Quasisymmetric Functions*, Ph.D. Thesis, University of Athens (2021).

[12] T. K. Petersen, Enriched P-partitions and peak algebras, *Adv. Math.* **209** (2007), 561–610.

[13] T. K. Petersen, Two-sided Eulerian numbers via balls in boxes, *Math. Mag.* **86** (2013), 159–176.

[14] T. K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts, Birkhäuser (2015).

[15] D. Rawling, Enumeration of permutations by descents, idescents, imajor index, and basic components, *J. Combin. Theory Ser. A* **36** (1984), 1–14.

[16] R. P. Stanley, *Enumerative Combinatorics*, Vol. I, 2nd ed., Cambridge University Press, Cambridge (2012).

[17] E. Steingrímsson, Permutation statistics of indexed permutations, *European J. Combin.* **15** (1994), 187–205.

[18] M. Visontai, Some remarks on the joint distribution of descents and inverse descents, *Electron. J. Combin.* **20** (2013), P52.

Received: 22.01.2024
Accepted: 25.03.2024

(1) School of Mathematics and Statistics, Chongqing University of Posts and Telecommunications,
Chongqing 400065, P. R. China
E-mail: `xqgao7@163.com`

(2) School of Mathematics and Statistics, Chongqing University of Posts and Telecommunications,
Chongqing 400065, P. R. China
Advanced Research Institute, Chongqing University of Posts and Telecommunications,
Chongqing 400065, P. R. China
E-mail: `zkli@cqupt.edu.cn`

(3) School of Mathematics and Statistics, Chongqing University of Posts and Telecommunications,
Chongqing 400065, P. R. China
E-mail: `llwan0@163.com`

(4) School of Mathematics and Statistics, Chongqing University of Posts and Telecommunications,
Chongqing 400065, P. R. China
Advanced Research Institute, Chongqing University of Posts and Telecommunications,
Chongqing 400065, P. R. China
E-mail: `yangyingxue@cqupt.edu.cn`