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On a problem in valuation theory

by

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Dedicated to the memory of our Teacher, dr. doc. Nicolae Popescu

Abstract

Let (K, v) be a nontrivial Krull valued number field and let $\overline{K} = \overline{\mathbf{Q}}$ be a fixed algebraic closure of K. We say that an extension L/K, $L \subset \overline{K}$, is a *v-extension* of (K, v) if v does not split in L.

In the 2000s dr. doc. Nicolae Popescu (Institute of Mathematics of the Romanian Academy) stated the following hypothesis:

"An algebraic number field K with a nontrivial Krull valuation v on it cannot have a normal v-maximal extension." Trying to solve this problem we considered a Krull valued number field (K, v) with some additional properties and we constructed a class of v-maximal extensions of K which are not normal extensions of K. Thus, in general, the above hypothesis is still open.

We also give an example of a nontrivial valued field (T, w) which has a normal w-maximal extension. Some other auxiliary results are given on these mysterious mathematical objects, v-maximal extensions.

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1 Introduction

Let (K, v) be a Krull nontrivial valued field and let $\overline{K}$ be a fixed algebraic closure of K. We say that (K, v) is a *Henselian field* if v has a unique extension $\overline{v}$ to $\overline{K}$. This property of (K, v) is an intrinsic property, that is, it does not depend on the fixed algebraic closure $\overline{K}$. If (K, v) is not Henselian, then there exists subfields L, $K \subset L \subsetneqq \overline{K}$ such that v has a unique extension u to L, and L is maximal with this last property, that is u splits in any proper extension N/L, $N \subset \overline{K}$. We call such an extension L/K, a *v-maximal extension.* A *v-extension* M/K is simply a field extension M of K so that v does not split in M. It is clear that a v-maximal extension of (K, v) is a maximal element in the set of all v-extensions of K (see also [1] and [2]). In the 2000s dr. doc. Nicolae Popescu (Institute of Mathematics of the Romanian Academy) made the following hypothesis (NP-hypothesis):

"An algebraic number field K with a nontrivial Krull valuation v on it cannot have a normal v-maximal extension."

In this note we describe two situations in which one can find non-normal v-maximal extensions for a Krull valued algebraic number field (K, v) (see Theorem 6 and Theorem

7). In Remark 3 we give an example of a Krull valued field, namely $(\mathbb{C}(X), v)$, which has a normal v-maximal extension. Here v is the usual X-adic valuation. Thus, the NP-hypothesis is not true for function fields.

So that everything is clear and easy to read, in the Section 2 we collect some known results useful in our study. Proposition 1 is a slightly modified result of R. Brauer ([7], pages 107-108).

In Section 3 we put all the main new results together with their proofs. Lemma 1 says that if L is a v-extension that is also a normal extension of K, then $Gal(L/K) \simeq Gal(LK^h/K^h)$, where K^h is a henselization of K. Proposition 2 supplies some sufficient conditions on (K, v) so that we can find a v-maximal extension of K which is not normal. In Theorem 5 we describe a general situation when the Galois group of a normal extension of a number field is not a solvable group. This result is the key point in the proof of Theorem 7. Theorem 6 says that a real number field K with a nontrivial Krull valuation v on it satisfies the conditions of Proposition 2, that is, in this case, we can find non-normal v-maximal extensions of (K, v). Theorem 7 says that any nontrivial Krull valued number field (K, v) satisfies the conditions of Proposition 2 if $K/\mathbb{Q}$ is normal and $Gal(K/\mathbb{Q})$ is a solvable group. Thus, in this last case, we can find v-maximal extensions of (K, v) which are not normal extensions.

2 Notation, definitions and some general results

Let (K, v) be a perfect nontrivial Krull valued field of rank 1 and let us fix a completion $(\widetilde{K}, \widetilde{v})$ of it. Let $\left(\overline{\widetilde{K}}, w\right)$ be a fixed algebraic closure of $\widetilde{K}$ and let w be the unique extension of $\widetilde{v}$ to $\overline{\widetilde{K}}$. Let $\overline{K}$ be the algebraic closure of K in $\overline{\widetilde{K}}$ and let $\overline{v}$ be the restriction of w to $\overline{K}$. In general, if L/K, $L \subset \overline{K}$, is an extension of K, we denote by v_L the restriction of $\overline{v}$ to L. It is easy to see that $\overline{K}$ is also an algebraic closure of K. The valuation $\overline{v}$ depends on $(\widetilde{K}, \widetilde{v})$ and on $\overline{\widetilde{K}}$, that is, it is unique up to continuous isomorphisms of valued fields and algebraic closed fields over K respectively. All the topological properties which follow are relative to the topology induced by this valuation $\overline{v}$ on $\overline{K}$ or to the one induced by w on $\overline{\widetilde{K}}$.

In general, a subfield L, $K \subset L \subset \overline{K}$, is called a *Henselian field relative to* $\overline{v}$ if $\overline{v}$ is the unique extension of v_L to $\overline{K}$, where v_L is the restriction of $\overline{v}$ to L. Since the construction of $(\widetilde{K}, \widetilde{v})$ and of $(\overline{\widetilde{K}}, w)$ are unique up to isomorphisms of fields, if L/K, $L \subset \overline{K}$, is a Henselian field relative to $\overline{v}$, then it is a Henselian extension in the usual sense (see [10], 5.1, Theorem 1, or [6], Theorem 2.40). Moreover, since Hensel's Lemma is true in $\widetilde{K}$ and since this lemma involves only algebraic quantities, we see that the algebraic part of $\widetilde{K}$, that is $K^h = \widetilde{K} \cap \overline{K}$ (the intersection is considered in $\overline{\widetilde{K}}$), is a Henselian extension of (K, v). Now, since the residue field of any subextension M/K of $\widetilde{K}$ is the residue field of K, we see that the Henselian field K^h (with respect to the restriction of $\widetilde{v}$ to K^h) is a least Henselian field which contains K (relative to the above constructed $\overline{v}$ on $\overline{K}$). K^h is called the *henselization* of (K, v) relative to $\overline{v}$. In fact, it depends on the fixed valuation $\overline{v}$ on $\overline{K}$. If we change $\overline{v}$ with another one, $\overline{v} \circ \sigma$, where $\sigma \in G = Gal(\overline{K}/K)$, then the henselization of (K, v) will be $\sigma^{-1}(K^h)$. In the following we work only with the standard valuation $\overline{v}$ constructed above, consequently K^h is unique. It is easy to see that K^h is the topological closure of K in $\overline{K}$

(relative to $\overline{v}$). It is also the algebraic closure of K in $\widetilde{K}$, the completion of K relative to v. Since K^h is a Henselian field, we see that any field L with $K^h \subset L \subset \overline{K}$ is also a Henselian field.

Lemma 1. *Keeping the above hypotheses, notation and definitions, let L/K be an extension of K in $\overline{K}$. Then $L^h = \widetilde{L} \cap \overline{K}$, the henselization of L with respect to $\overline{v}$, is equal to LK^h, the least subfield of $\overline{K}$ which contains L and K^h.*

Proof. First of all we want to prove that LK^h is the topological closure of L in $\overline{K}$. Since K^h is a Henselian field, we see that LK^h is also a Henselian field. Let α be an element in $\overline{K}$ which is in the topological closure of LK^h, that is, there exist a sequence $\alpha_n \to \alpha$ with $\alpha_n \in LK^h$. Krasner's Lemma says that there exists an $\alpha_N \in LK^h$ such that $LK^h[\alpha] \subset LK^h[\alpha_N] = LK^h$ ([10], 5.1, I, or [3], Ch. 2, Theorem 8). Hence, $\alpha \in LK^h$, that is, LK^h is topologically closed in $\overline{K}$. Since $L \subset LK^h$, we see that the topological closure of L in $\overline{K}$ is contained in LK^h. Since K^h is the topological closure of K in $\overline{K}$, we see that K^h is contained in the topological closure of L in $\overline{K}$. Thus LK^h is equal to the topological closure of L in $\overline{K}$. ∎

Let $G(\overline{v}) = Gal\left(\overline{K}/K^h\right) = \left\{\sigma \in Gal\left(\overline{K}/K\right) : \overline{v} \circ \sigma = \overline{v}\right\}$ be the *decomposition (splitting) group of* $\overline{v}$. It is a closed subgroup of $Gal\left(\overline{K}/K\right)$ with respect to the Krull topology (see [1], [6]). Moreover, it is also equal to $Gal_{cont,\overline{v}}(\overline{K}/K)$, the subgroup of all K-automorphisms of $\overline{K}$ which are continuous relative to the topology induced by $\overline{v}$. The inclusion $G(v) \subset Gal_{cont,\overline{v}}(\overline{K}/K)$ is obvious. Let σ be in $Gal_{cont,\overline{v}}(\overline{K}/K)$. We see that $\{x \in \overline{K} : \overline{v}(x) < 1\} = \{x \in \overline{K} : (\overline{v} \circ \sigma)(x) < 1\}$, that is, $\overline{v}$ and $\overline{v} \circ \sigma$ generate the same topology on $\overline{K}$ and thus, $\overline{v} \circ \sigma = s\overline{v}$, with s a positive real number (see [8], Proposition 3.3). Since $(\overline{v} \circ \sigma)(x) = \overline{v}(x)$ for any $x \in K$, we see that $s = 1$, that is, $\sigma \in G(v)$.

Definition 1. *An extension of fields L/K, $L \subset \overline{K}$, is a v-extension if v can be uniquely extended to L, that is, if v_L, the restriction of $\overline{v}$ to L, is the unique valuation on L which extends v, or if v does not split in L. A v-extension L is a v-maximal extension if it is maximal as a v-extension* (*see also [1] and [2]*).

Remark 1. *It is easy to see that L/K, $L \subset \overline{K}$ is a v-extension if and only if any K embedding $\sigma : L \to \overline{K}$ is continuous with respect to $\overline{v}$, i.e. $\left(\overline{v} \circ \overline{\sigma}\right)_L = v_L$ for any extension $\overline{\sigma}$ of σ to $\overline{K}$.*

The following three results are known, but we decided to give them in a more appropriate form for our purpose.

Theorem 1. (*[8], pages 161-167, or [5], Theorem 1.1*) *Let (K, v), $(\widetilde{K}, \widetilde{v})$, $(\overline{\widetilde{K}}, w)$ and $(\overline{K}, \overline{v})$ be as above, let L/K, $L \subset \overline{K}$ be an extension of fields and let u be another valuation on L which extends v. Then there exists an automorphism $\sigma \in G = Gal(\overline{K}/K)$ such that $u = (\overline{v} \circ \sigma)_L$. Moreover, if $\overline{v} \circ \sigma_1 = \overline{v} \circ \sigma_2$, with $\sigma_1, \sigma_2 \in G$, then $\sigma_1 \circ \sigma_2^{-1} \in G(\overline{v}) = Gal(\overline{K}/K^h)$.*

Theorem 2. (*[8], Proposition 8.2, or [6], Theorem 3.7*) *With the above notation and hypotheses, let α be an element in $\overline{K}$ and let $f_\alpha(X) \in K[X]$ be its minimal polynomial over K. Let $f_\alpha(X) = f_1^{n_1}(X) \cdots f_r^{n_r}(X)$ be the factorization of $f_\alpha(X)$ into distinct irreducible factors over K^h, the henselization of K with respect to $\overline{v}$. Let $\alpha_i \in \overline{K}$ be a fixed root of* *the polynomial* $f_i(X) \in K^h[X]$ *and let* $\sigma_i \in G = Gal(\overline{K}/K)$ *be such that* $\sigma_i(\alpha) = \alpha_i$, $i = 1, 2, ..., r$. *Then* $v_i = (\overline{v} \circ \sigma_i)_{K[\alpha]}$ *are all the distinct valuations on* $K[\alpha]$ *that extend* v *to* $K[\alpha]$. *Moreover, the completion* $\widetilde{K[\alpha]}_{v_i}$ *of* $K[\alpha]$ *relative to* v_i *is isomorphic to* $\widetilde{K}[\alpha_i]$, $i = 1, 2, ..., r$, *where* $\widetilde{K}$ *is the completion of* K *with respect to* v. *The statements remain true if we change* K^h *with* $\widetilde{K}$.

Theorem 3. (*[10], 5.1, G, or [6], Corollary 2.33) With the above notation and hypotheses, we assume that* (K, v) *is a Henselian valued field and let*

$$f(X) = a_0 + a_1X + ... + a_{n-1}X^{n-1} + X^n \in K[X]$$

be a monic irreducible polynomial over K. *Let* $\lambda_1 \in \overline{K}$ *be a fixed root of* f. *Then there exists a real number* $M > 0$ *such that if*

$$g(X) = b_0 + b_1X + ... + b_{n-1}X^{n-1} + X^n \in K[X]$$

with $v(a_i - b_i) > M$, $i = 0, 1, ..., n-1$, *then* g *is irreducible over* K *and it has at least one root* $\mu_1 \in \overline{K}$ *such that* $K[\lambda_1] = K[\mu_1]$. *Moreover, if* $\lambda_1, ..., \lambda_n$ *are all the roots of* $f(X)$ *and* $\mu_1, ..., \mu_n$ *are the roots of* $g(X)$, *then, for any* $N > 0$, *there exists another real number* $M_N > 0$ *such that if* $v(a_i - b_i) > M_N$, *then* $\overline{v}(\lambda_i - \mu_{\sigma(i)}) > N$, *where* σ *is a permutation of the set* $\{1, 2, ..., n\}$.

Corollary 1. *With the above notation and hypotheses,* $\overline{K}$ *is dense in* $\overline{\widetilde{K}}$ *with respect to* w, *the unique extension of* $\widetilde{v}$ *to* $\overline{\widetilde{K}}$ *and* $\overline{\widetilde{K}} = \widetilde{K}\overline{K}$. *Moreover,* $G(\overline{v}) = Gal\left(\overline{K}/K^h\right) \simeq Gal\left(\overline{\widetilde{K}}/\widetilde{K}\right)$, *this last isomorphism being a topological isomorphism relative to the Krull topologies. In particular, if the residue field of* (K, v) *is finite, then* $G(\overline{v})$ *is a prosolvable group (an inverse limit of solvable groups), because Gal* $\left(\overline{\widetilde{K}}/\widetilde{K}\right)$ *is a prosolvable group (see [11], 3-6-6).*

Proof. Let $\alpha \in \overline{\widetilde{K}}$ be an element of the algebraic closure $\overline{\widetilde{K}}$ of $\widetilde{K}$, the fixed completion of K with respect to v. Let $f_\alpha(X) \in \widetilde{K}[X]$ be its minimal polynomial over $\widetilde{K}$. Let $M > 0$ be a sufficiently large positive real number and let $g(X) \in K[X]$ be a monic irreducible polynomial with coefficients sufficiently close to the corresponding coefficients of $f_\alpha(X)$ such that a root $\beta \in \overline{K}$ of $g(X)$ verifies the inequality: $\overline{v}(\alpha - \beta) > M$ and $\widetilde{K}[\alpha] = \widetilde{K}[\beta]$ (see Theorem 3). Thus, $\overline{K}$ is dense in $\overline{\widetilde{K}}$ and $\overline{\widetilde{K}} = \widetilde{K}\overline{K}$.

Let now $\sigma \in Gal\left(\overline{K}/K^h\right)$. Then $\overline{v} \circ \sigma = \overline{v}$, since $\overline{v}$ is the unique extension to $\overline{K}$ of the restriction $\overline{v}_{K^h}$ of $\overline{v}$ to K^h. So σ is continuous with respect to $\overline{v}$. Since $\overline{K}$ is dense in $\overline{\widetilde{K}}$, this σ can be uniquely extended (by continuity) to a $\widetilde{K}$-automorphism $\widetilde{\sigma}$ of $\overline{\widetilde{K}}$ over $\widetilde{K}$. Now, it is easy to see that $\sigma \to \widetilde{\sigma}$ is a topological isomorphism between $Gal\left(\overline{K}/K^h\right)$ and $Gal\left(\overline{\widetilde{K}}/\widetilde{K}\right)$. It is not difficult to conclude (see [11], Proposition 3-6-6) that, if the residue field of (K, v) is finite, then $Gal\left(\overline{\widetilde{K}}/\widetilde{K}\right)$ is a prosolvable group. Thus $Gal\left(\overline{K}/K^h\right)$ is also a prosolvable group. ∎

We need now to make some appropriate changes in the statement of Theorem 8, Ch. II, [7].

Proposition 1. *Let $M \geq 5$ be a real number. Then, there exists an Eisenstein polynomial $f(X) \in \mathbb{Z}[X]$ relative to a given prime number p, of a given prime degree $q \geq M$ such that the equation $f(x) = 0$ has exactly two non-real roots. In this last case, the Galois group of f over $\mathbb{Q}$ is isomorphic to the symmetric group S_q. Thus, the Galois group of f over $\mathbb{Q}$ is not a solvable group.*

Proof. We use here an idea of R. Brauer (see [7], pages 107, 108). Let $m > 0$ be a positive integer that will be fixed later, and let $n_1 < n_2 < ... < n_{q-2}$ be $q-2$ integers. We define the polynomial $g(X) \in \mathbb{Z}[X]$:

$$g(X) = (X^2 + m)(X - pn_1) \cdot ... \cdot (X - pn_{q-2}).$$

The equation $g(x) = 0$ has exactly $q-2$ real roots and $q-3$ local (relative) extremum points, each of them situated in the intervals (pn_j, pn_{j+1}), $j = 1, 2, ..., q-3$. Since $pn_{j+1} - pn_j > p$, each such interval contains an integer of the form ph_j, where $h_j \in \mathbb{Z}$. It is easy to see that $|g(ph_j)| > p$, so the equation $f(x) = g(x) - p = 0$ has at least $q-3$ real distinct roots each of them in one of the intervals (pn_j, pn_{j+1}), $j = 1, 2, ..., q-3$. Moreover, since $f(pn_{q-2}) = -p$ and $f(\infty) = \infty$, there is another root in the interval (pn_{q-2}, ∞). Thus, $f(x) = 0$ has at least $q-2$ real roots. Let $\alpha_1, ..., \alpha_q$ be all the roots in $\mathbb{C}$ of the equation $f(x) = 0$. Since

$$\sum_{i=1}^{q} \alpha_i = p \sum_{i=1}^{q-2} n_i, \quad \sum_{i<j} \alpha_i \alpha_j = m + p^2 \sum_{i<j} n_i n_j,$$

we get that $\sum_{i=1}^{q} \alpha_i^2 = p^2 \sum_{i=1}^{q-2} n_i^2 - 2m$. We see that if we take m a multiple of p and $m > \frac{1}{2} p^2 \sum_{i=1}^{q-2} n_i^2$, not all roots can be real roots. In this case, since $q-2$ roots are real, we have to conclude that the equation $f(x) = 0$ has exactly two non-real roots, conjugate one to each other. Moreover, it is easy to see that the polynomial

$$f(X) = (X^2 + m)(X - pn_1) \cdot ... \cdot (X - pn_{q-2}) - p \in \mathbb{Z}[X]$$

is an Eisenstein polynomial relative to p. Thus it is irreducible over $\mathbb{Q}$.

Let us assume that $\alpha_1, \alpha_2 = \overline{\alpha}_1$ are the complex roots of f. Then the restriction $\overline{e}$ of the complex conjugation to $\mathbb{Q}_f$, the splitting field of f (over $\mathbb{Q}$), $\overline{e}(\alpha_1) = \alpha_2$, $\overline{e}(\alpha_2) = \alpha_1$, and $\overline{e}(\alpha_j) = \alpha_j$ for any $j = 3, 4, ..., q$, is a transposition in the Galois group $G = Gal(\mathbb{Q}_f/\mathbb{Q}) \subset S_q$. Since the prime number q divides the order of G, there exists in G a cyclic Sylow subgroup of order q. This last one is generated by a cycle of order q in S_q. Since G contains a transposition and a cycle of order q, it must be equal to S_q (see [7], pag.106). □

Remark 2. *Let K be a real number field such that the Eisenstein polynomial $f \in \mathbb{Q}[X]$ from Proposition 1 is also irreducible over K. Let K_f be the splitting field of f over K. Since $q = \deg_K f$ is a prime number, the Galois group $G_1 = Gal(K_f/K)$ contains a Sylow subgroup of order q, which is a cyclic subgroup of G_1. Moreover, the conjugation morphism $\overline{e} : \alpha \to \overline{\alpha}$, restricted to K, gives rise to a transposition in $G_1 \subset S_q$. Thus, $G_1 = S_q$(see [7], pages. 106). Since $q \geq 5$, we see that G_1 is not a solvable group.*

3 v-normal extensions

We preserve the notation, definitions and hypotheses introduced in Section 2.

Definition 2. *A v-extension L/K, $L \subset \overline{K}$, is called a v-normal extension if L is also a normal extension of K.*

Problem. Can a v-normal extension of K be a v-maximal extension?

In [1], Remark 3.2, we considered the valued field $(\mathbb{Q}, v_2)$, where v_2 is the 2-adic valuation on $\mathbb{Q}$ and the field $L_1 = \mathbb{Q}[\sqrt[7]{2}]$. It is clear that v_2 does not split in L_1, so $L_1/\mathbb{Q}$ is a v_2-extension. There we have proved that any v_2-maximal extension of $\mathbb{Q}$ that contains L_1 cannot be a normal extension of $\mathbb{Q}$.

Lemma 2. *With the above hypotheses, notation and definitions, let $L_1 \subset \overline{K}$ be a finite normal extension of K which is also a v-extension of (K, v). Then $Gal\,(L_1/K) = Gal\left(L_1K^h/K^h\right)$. This means that for a finite v-normal extension L of K one has $Gal(L/K) = Gal(L^h/K^h)$ (see Lemma 1).*

Proof. Let $\gamma \in \overline{K}$ such that $L_1 = K[\gamma]$ and let $f_\gamma \in K[X]$ be the minimal polynomial of γ over K. Since v does not split in L_1, $f_\gamma \in K^h[X]$ is also the minimal polynomial of γ over K^h (see Theorem 2). Since $L_1K^h = K^h[\gamma]$, we see that $Gal\,(L_1/K)$ and $Gal\left(L_1K^h/K^h\right)$ are the Galois groups of one and the same irreducible polynomial f_γ. Thus they are equal. More exactly, we have in general that $Gal\,(L_1/K) \supset Gal\left(L_1K^h/K^h\right)$. Since $\deg_K f_\gamma = \deg_{K^h} f_\gamma$, these last Galois groups must be equal. ∎

The next theorem appears in [2], Theorem 6. For reader convenience, we give it here again with a new proof.

Theorem 4. *With the above notation and hypotheses, let L/K be be an extension in $\overline{K}$. Then, L/K is a v-maximal extension of (K, v) if and only if L/K is a v-extension and L is dense in $\overline{K}$ relative to the valuation $\overline{v}$.*

Proof. $\Rightarrow$) We assume that L/K is a v-maximal extension. The topological closure of L in $\overline{K}$ is $LK(v)$, where $K(v) = K^h$, the henselization of (K, v) relative to $\overline{v}$ (Lemma 1). It is sufficient to prove that $LK(v) = \overline{K}$. Let α be in $\overline{K}$ and let $g_\alpha \in LK(v)[X]$ be its minimal polynomial over $LK(v)$. Since $LK(v)$ is a henselian field, we see that g_α is also irreducible in $LK(v)(\alpha)$, so $\deg g_\alpha = 1$ over $LK(v)$, that is $\alpha \in LK(v)$.

$\Leftarrow$) We assume now that L/K is a v-extension and L is dense in $\overline{K}$. Since L is dense in $\overline{K}$, its henselization $LK(v)$ is exactly $\overline{K}$. Suppose that L/K is not a v-maximal extension. Thus, there exists $\beta \in \overline{K}$ such that v_L, the unique extension of v to L decomposes in at least two extensions in $L[\beta] \supsetneq L$. Thus, the minimal polynomial g_β of β over L, is not irreducible over $L[\beta]$, but it is irreducible over the henselization $LK(v)(v) = \overline{K}$, a contradiction. Hence L/K is a v-maximal extension. ∎

In the following we keep the above hypotheses, notation and definitions.

Proposition 2. *Let (K, v) be a perfect nontrivial Krull valued field of rank 1 with a finite residue field. We assume that there exists a finite extension K_1/K $(K_1 \subset \overline{K})$ such that v does not split in K_1, i.e. K_1 is a v-extension of K. We also assume that the Galois group $Gal(L_1/K)$ of the least normal extension L_1 which contains K_1 is not a solvable group. Let K_2 be any v-maximal extension of K which contains K_1. Then K_2 cannot be a normal extension of K.*

Proof. We assume that K_2/K is a normal extension. Since K_2 contains K_1 and L_1 is a normal extension, we get that $L_1 \subset K_2$. Let $\gamma \in L_1$ such that $L_1 = K[\gamma]$ and let f_γ be the minimal polynomial of γ over K. Since v does not split in L_1, f_γ is also a minimal polynomial of γ over $K^h = \widetilde{K} \cap \overline{K}$, the henselization of K (see Theorem 2). So $Gal\left(L_1K^h/K^h\right) = Gal(L_1/K)$ (see Lemma 2) is not a solvable group. But $Gal\left(L_1K^h/K^h\right)$ is a finite group factor of $Gal\left(\overline{K}/K^h\right)$ which is a prosolvable group (see Corollary 1). Thus $Gal\left(L_1/K\right)$ is a solvable group, a contradiction. Hence K_2 cannot be a normal extension of K. □

Theorem 5. *Let K be a finite extension of $\mathbb{Q}$, the rational number field, and let v be a nontrivial valuation on K. Let v_p be the restriction of v to $\mathbb{Q}$ and let q be a prime number with $q > [K : \mathbb{Q}]$ and $q \geq 5$. Let $\mathbb{Q}_f$ be the splitting field of f over $\mathbb{Q}$, where*

$$f(X) = X^q + a_{q-1}X^{q-1} + ... + a_0 \in \mathbb{Z}[X] \tag{3.1}$$

is an Eisenstein polynomial relative to p so that $Gal(\mathbb{Q}_f/\mathbb{Q}) \simeq S_q$, the symmetric group of all permutations with q elements (see Proposition 1). Then f is irreducible over $(\widetilde{K}, \widetilde{v})$, the completion of (K, v), that is, v does not split in $K_1 = K[\alpha]$ for any root α of f. Moreover, if L_1 is the splitting field of f over K, then $Gal(L_1/\mathbb{Q})$ is not a solvable group.

Proof. Let e be the ramification index of v relative to v_p, i.e. $v(p) = e$. Since $e \leq [K : \mathbb{Q}] < q$ and since q is a prime number, we see that $(e, q) = 1$. Moreover, $v(a_0) = v(p) = e \leq v(a_i)$, $0 \leq i \leq q-1$, that is, $v(a_i) \geq e > \frac{ie}{q}$, $i = 1, 2, ..., q-1$. Thus we can apply a generalization of the Eisenstein criterion (see [9], Proposition 2.2) for the local field $(\widetilde{K}, \widetilde{v})$, the completion of K relative to v, and find that f is irreducible over $\widetilde{K}$. In particular, it is irreducible over K. Let now $L_1 = K_f$ be the splitting field of f over K and let $K_1 = K[\alpha]$, where α is a fixed root of f in $\overline{\mathbb{Q}}$. The valuation v does not split in K_1 because f is irreducible over $\widetilde{K}$ (see Theorem 2). Since $\mathbb{Q} \subset \mathbb{Q}_f \subset L_1$ and

$$Gal(L_1/\mathbb{Q})/Gal(L_1/\mathbb{Q}_f) \simeq Gal(\mathbb{Q}_f/\mathbb{Q}) \simeq S_q, q \geq 5,$$

we see that $Gal(L_1/\mathbb{Q})$ cannot be a solvable group. □

Theorem 6. *Let (K, v) be a real number field with a nontrivial Krull valuation v on K. Then there exists a non-normal v-maximal extension of K.*

Proof. Let us take an Eisenstein polynomial f over K like in Theorem 5 (see formula (3.1)) and let us fix a root $\alpha \in \overline{\mathbb{Q}}$ of f. Denote $K_1 = K[\alpha]$ and $L_1 = K_f$, the splitting field of f over K. Thus K_1/K is a v-extension and $Gal(L_1/K)$ is not a solvable group (see Remark

2). Now, we can take any v_{K_1}-extension K_2 of K_1 (that is also a v-extension of K), and apply Proposition 2 to see that K_2 is not a normal extension of K. Here v_{K_1} is the unique extension of v to K_1. $\square$

Theorem 7. *Let K be a finite normal extension of $\mathbb{Q}$, and let v be a nontrivial Krull valuation on K. We also assume that $Gal(K/\mathbb{Q})$ is a solvable group. Then there exists a non-normal v-maximal extension of K.*

Proof. Let us take an Eisenstein polynomial f over K like in Theorem 5, let $K_1 = K[\alpha]$, where α is a root of f, and let $L = K_f$ be the splitting field of f over K. Since $Gal(\mathbb{Q}_f/\mathbb{Q}) \simeq S_q$, $q \geq 5$ is not solvable, we see that $Gal(L/\mathbb{Q})$ is not a solvable group (see Theorem 5). From $\mathbb{Q} \subset K \subset L$ and from

$$Gal(L/\mathbb{Q})/Gal(L/K) \simeq Gal(K/\mathbb{Q}),$$

we see that $Gal(L/K)$ cannot be a solvable group, because $Gal(K/\mathbb{Q})$ is a solvable group and $Gal(L/\mathbb{Q})$ is not. Now, we can take a v_{K_1}-maximal extension K_2 of K, which contains K_1, that is also a v-maximal extension of K, and apply Proposition 2 to finally find that K_2 is not a normal extension of K. $\square$

Remark 3. *Let $K = \mathbb{C}(X)$, be the rational function field over the complex number field. Let v be the X-adic valuation on K and let $(\widetilde{K} = \mathbb{C}((X)), \widetilde{v})$ be its X-adic completion. We know (see [4], Ch. IV, Theorem 6) that any finite extension K' of $\widetilde{K}$, $[K' : \widetilde{K}] = n$, is of the form: $K' = \widetilde{K}[X^{\frac{1}{n}}]$. Thus, the algebraic closure $\overline{\widetilde{K}}$ of $\widetilde{K}$ is the field $\overline{\widetilde{K}} = \cup_{n=1}^{\infty} \mathbb{C}((X))[X^{\frac{1}{n}}]$ of all Puisseaux series with complex coefficients. For $L = \cup_{n=1}^{\infty} \mathbb{C}(X^{\frac{1}{n}})$, we see that $\mathbb{C}((X))L = \overline{\widetilde{K}}$. Thus, L is dense in $\overline{\widetilde{K}}$ with respect to the unique extension of $\widetilde{v}$ to $\overline{\widetilde{K}}$. So L is a v-maximal extension of (K, v) (see Theorem 4, or [2], Theorem 6). Moreover, it is also a normal extension of K. Hence, we have here an example of a v-maximal extension which is normal at the same time.*

Remark 4. *Theorem 6 and Theorem 7 supply us with a large class of effective examples of number fields (K, v) with Krull valuations v on them so that they admit many non-normal v-maximal extensions. For instance, if $g \in \mathbb{Q}[X]$ is a polynomial of an odd degree, and γ is a real root of it, we can take $K = \mathbb{Q}[\gamma]$ and for v we can take any extension of a p-adic valuation v_p of $\mathbb{Q}$, and then apply Theorem 6. In order to apply Theorem 7 we can take any quadratic extension $K = \mathbb{Q}[\sqrt{d}]$, where $d \in \mathbb{Q}$ and $\sqrt{d} \notin \mathbb{Q}$, and for v we can consider any extension to K of a p-adic valuation v_p on $\mathbb{Q}$. The abundance of non-normal v-maximal extensions can also be seen from the variety of the Eisenstein polynomials f of the form (3.1).*

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