

Transcendence criteria for non quadratic Ruban continued fractions in \mathbb{Q}_p

by

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Abstract

The central objective of this work is to establish a new transcendence criterion for non quadratic p -adic numbers by using their Ruban continued fractions. By taking a pair (α, α') of p -adic numbers and under certain combinatorial conditions, we prove that one of α and α' is transcendental or both are quadratic.

Key Words: Ruban continued fractions, p -adic numbers, subspace theorem, transcendence.

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1 Introduction

The notion of a continued fraction was introduced in multiple research works of several mathematicians in order to provide results on the transcendence of a real number and to approximate it by rationals. As far as this issue is concerned, we have numerous results. In particular, Maillet [15], in 1906, provided illustrative examples of transcendental continued fractions with bounded partial quotients. Another result was found by Baker [8], who proved equally that there are many continued fractions with bounded partial quotient which converge to transcendental numbers. Motivated and inspired by Baker's finding and using the Subspace theorem [20], Adamczewski and Bugeaud [1] made further important discoveries in this direction. Subsequently, they gave in [2] a criterion about palindromic continued fractions and proved that if the sequence of partial quotients of an irrational number α begins with arbitrary long palindromes, then α is either quadratic or transcendental. Next in [3], Adamczewski and Bugeaud established an extension of this criterion. Relying on the Schmidt Subspace Theorem [20], they studied a pair (α, α') of real numbers and proved, under certain conditions on their continued fraction expansion, that either one of α and α' is transcendental, or both are in the same real quadratic field. Before introducing the result of Adamczewski and Bugeaud in [3], we present some practical properties of the terminology from combinatorics on words.

Let \mathcal{A} denote a countable set. The length of a finite word W on the alphabet \mathcal{A} , that is the number of letters composing W , is denoted by $|W|$. The reverse word of $W = a_1 \dots a_n$ is the word $\overline{W} = a_n \dots a_1$. We say that, W is a palindrome if and only if $W = \overline{W}$. We identify any sequence $\mathbf{a} = (a_n)_{n \geq 1}$ of elements from \mathcal{A} with the infinite word $a_1 a_2 \dots a_n \dots$

Theorem 1. ([3], Th. 1.4)

Let $\mathbf{a} = (a_n)_{n \geq 1}$ and $\mathbf{a}' = (a'_n)_{n \geq 1}$ be two sequences of positive integers. Set

$$\alpha = [0; a_1, a_2, \dots], \quad \alpha' = [0; a'_1, a'_2, \dots].$$

We assume that there exist two sequences of finite words $(U_n)_{n \geq 1}$ and $(V_n)_{n \geq 1}$ with the alphabet $\mathbb{Z}_{\geq 0}$ such that:

- i) for every $n \geq 1$, the word V_n is a prefix of the word \mathbf{a} ;
- ii) for every $n \geq 1$, the word $U_n \bar{V}_n$ is a prefix of the word \mathbf{a}' , where \bar{V}_n is the reverse word of V_n ;
- iii) the sequence $\left(\frac{|U_n|}{|V_n|}\right)_{n \geq 1}$ is upper bounded;
- iv) the sequence $(|V_n|)_{n \geq 1}$ is increasing.

Let us now denote $\left(\frac{p_n}{q_n}\right)_{n \geq 1}$ the sequence of convergents to α' and let us assume that the sequence $(q_n^{\frac{1}{n}})_{n \geq 1}$ is bounded. Then, either one of α and α' is transcendental, or both are in the same real quadratic field.

Recently, in 2013, Y. Bugeaud [9] has developed a transcendence criterion for quasi-palindromic continued fractions by using the subspace theorem.

The theory of real continued fractions has been generalized for the field of p -adic numbers \mathbb{Q}_p . The first idea in this way appeared in [14]. Later, an improvement was made by Schneider [21] who set forward a first definition for a p -adic continued fraction expansion. Two years later, Ruban gave in [18] a different, a simpler and a more similar algorithm to the real case. However, contrarily to the real case, these two algorithms provide finite or periodic continued fractions for rational numbers. In this respect, Wang [22] and Laohakosol [12] proved independently that a p -adic number α is rational if and only if its Ruban continued fraction is finite or ultimately periodic with all partial quotients in their period is equal to $(p - p^{-1})$. Furthermore, for the quadratic p -adic numbers, Ooto [16] confirmed that an analogue of Lagrange's theorem does not exist. Moreover, Capuano, Veneziano and Zannier [10] reported an effective condition to detect the periodicity of p -adic numbers. Using the properties of Ruban's continued fraction and motivated by the same type of research in the real case, multiple authors reported pertinent discoveries. In 2017, inspired by a Baker's result, Ooto [16] gave a criterion of transcendence for quasi-periodic continued fractions in the field of p -adic numbers. This result was based on a theorem of Ridout [17]. Motivated by a result of Adamczewski and Bugeaud [3], the authors obtained in [7, 6] new transcendence criteria for p -adic continued fractions. They proved that some palindromic and quasi-palindromic p -adic continued fractions, under some conditions, converge to transcendental numbers. Subsequently, the authors asserted in [4] that quasi-periodic p -adic continued fractions converge to transcendental numbers (under some additional conditions).

As far as this paper is concerned, a new criterion of transcendence for Ruban p -adic continued fractions is given with which an analogous of Theorem 1 stated above is proved by using the p -adic version of the Schmidt Subspace Theorem, introduced by Schlickewei [19]. In fact, elements of a sequence of Ruban continued fraction are rationals in

$\mathbb{Z} \left[\frac{1}{p} \right] \cap (0, p)$ and not integers like elements of a sequence of real continued fractions. Unlike the real case, the p -adic version of the Subspace Theorem combines archimedean and non-archimedean absolute values.

The current work is organized as follows: In Section 2, we recall some notation and results for Ruban continued fractions in \mathbb{Q}_p , and we equally identify certain basic proper-

ties which are necessary in our work. Our results of transcendence of p -adic numbers are displayed in Section 3. Subsequently, in Section 4 we provide proofs for our main theorems as well as for some necessary lemmas.

2 Ruban continued fractions

Let p be a prime number. The field of p -adic numbers, \mathbb{Q}_p , is the completion of \mathbb{Q} with respect to the p -adic absolute value $|\cdot|_p$ normalized such that $|p|_p = p^{-1}$. As such it is analogous to the field of real numbers with respect to the usual absolute value $|\cdot|_\infty$. It is worth noting that \mathbb{Q}_p is also the fraction field of the p -adic integers ring \mathbb{Z}_p where:

$$\mathbb{Z}_p = \left\{ \sum_{i=0}^{+\infty} c_i p^i; | c_i \in \{0, \dots, p-1\} \right\},$$

and so,

$$\mathbb{Q}_p = \left\{ \sum_{i=k}^{+\infty} c_i p^i; | c_i \in \{0, \dots, p-1\}; k \in \mathbb{Z} \right\}.$$

Recall that the p -adic absolute value $|\cdot|_p$ over \mathbb{Q}_p is defined for any $\alpha \in \mathbb{Q}_p$:

$$|\alpha|_p = \begin{cases} 0 & \text{for } \alpha = 0; \\ p^{-\nu_p(\alpha)} & \text{for } \alpha \neq 0, \end{cases}$$

where ν_p is the p -adic valuation: $\nu_p : \mathbb{Q}_p \rightarrow \mathbb{Z} \cup \{+\infty\}$,

$$\nu_p(\alpha) = \begin{cases} +\infty, & \text{if } \alpha = 0, \\ \inf\{i / c_i \neq 0\}, & \text{if } \alpha = \sum_{i=k}^{+\infty} c_i p^i \text{ where } c_i \in \{0, \dots, p-1\}. \end{cases}$$

For every p -adic number $\alpha = \sum_{i=k}^{+\infty} c_i p^i$. Set the p -adic floor part of α as:

$$[\alpha]_p = \begin{cases} b_{-k} p^{-k} + b_{-k+1} p^{-k+1} + \dots + b_{-1} p^{-1} + b_0, & \text{if } k \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

and $\{\alpha\}_p = b_1 p + b_2 p^2 + \dots + b_n p^n + \dots$ as the fractional part of α . We write $a_0 = [\alpha]_p$. If $\alpha \neq [\alpha]_p$, then α can be written in the form :

$$\alpha = \alpha_0 = a_0 + \frac{1}{\alpha_1},$$

with $\alpha_1 \in \mathbb{Q}_p$ and $|\alpha_1|_p \geq p$. If $\alpha_1 \neq [\alpha_1]_p$, then α_1 can be written as:

$$\alpha_1 = [\alpha_1]_p + \frac{1}{\alpha_2},$$

with $\alpha_2 \in \mathbb{Q}_p$. If $\alpha_n \neq [\alpha_n]_p$, we continue by writing $\alpha_n = [\alpha_n]_p + \frac{1}{\alpha_{n+1}}$. If $\alpha_n = [\alpha_n]_p$, we stop this process.

Finally, every p -adic number α has a unique Ruban continued fraction in the following form:

$$\alpha = [\alpha_0]_p + \frac{1}{[\alpha_1]_p + \frac{1}{\ddots + \frac{1}{[\alpha_{n-1}]_p + \frac{1}{a_n + \frac{1}{\ddots}}}}}$$

where $a_k = [\alpha_k]_p$ and $\alpha_k = \frac{1}{\alpha_{k-1} - a_{k-1}}$, for all $k \geq 1$.

Now, for an infinite Ruban continued fraction $\alpha = [a_0, a_1, \dots]_p$, we define some positive rational numbers p_n, q_n by using the following recurrence equations:

$$\begin{cases} p_{-1} = 1, & p_0 = a_0, & q_{-1} = 0, & q_0 = 1 \\ p_n = a_n p_{n-1} + p_{n-2}, & q_n = a_n q_{n-1} + q_{n-2}, & \text{for any } n \geq 1. \end{cases}$$

The Ruban continued fraction has several similar properties with the real continued fractions. Notably, $\frac{p_n}{q_n}$ is called the n^{th} convergent of α and we have this following proposition.

Proposition 1. [16]

$$\left| \alpha - \frac{p_n}{q_n} \right|_p < |q_n|_p^{-2} \text{ and } \lim_{n \rightarrow \infty} \frac{p_n}{q_n} = \alpha$$

3 Results

Throughout this section, for a given bounded sequence $(a_i)_{i \geq 1} \in \mathbb{Q}_+^{\mathbb{N}} x$, we denote

$$A = \max\{a_i, i \geq 1\} \text{ and } B = B(A) = \frac{A + \sqrt{A^2 + 4}}{2}.$$

Our first result in this section is the following theorem which gives a new criterion of transcendence for Ruban continued fractions.

Theorem 2. Let p be a prime number such that $p > B^3$ and let $\alpha = [0; a_1, a_2, \dots, a_i, \dots]_p$ be a p -adic number represented as a Ruban continued fractions, where the sequence $(a_i)_{i \geq 1}$ is not ultimately periodic and $\{-\nu_p(a_i)\}_{i \geq 1} x$ is bounded. Let $\left(\frac{r_n}{s_n}\right)_{n \geq 1}$ be the sequence of convergents of Ruban continued fraction expansion of α . If there exist a non-zero integer quadruple (a, b, c, d) and an infinite set \mathcal{N} of distinct positive integers such that

$$a s_{n-1} + b r_{n-1} + c s_n + d r_n = 0, \forall n \in \mathcal{N}, \quad (3.1)$$

then α is either quadratic or transcendental.

Our next statement deals with a class of continued fractions satisfying the following condition denoted by $(*)_\omega$.

Let $\mathbf{a} = (a_i)_{i \geq 1}$ and $\mathbf{a}' = (a'_i)_{i \geq 1}$ be two sequences of elements from \mathcal{A} , that we identify with the infinite words $a_1 a_2 \dots$ and $a'_1 a'_2 \dots$, respectively. We assert that the pair $(\mathbf{a}, \mathbf{a}')$ satisfies Condition $(*)_\omega$ if there exist two sequences of finite words $(U_n)_{n \geq 1}$ and $(V_n)_{n \geq 1}$ such that:

- (i) for every $n \geq 1$, the word V_n is a prefix of the word \mathbf{a} ;
- (ii) for every $n \geq 1$, the word $U_n \bar{V}_n$ is a prefix of the word \mathbf{a}' ;
- (iii) the sequence $\left(\frac{|U_n|}{|V_n|} \right)_{n \geq 1}$ is upper bounded by $\omega > \frac{1}{2}$;
- (iv) the sequence $(|V_n|)_{n \geq 1}$ is increasing.

Theorem 3. *Let p be a prime number. Let $\alpha = [0; a_1, a_2, \dots, a_i, \dots]_p$ and $\alpha' = [0; a'_1, a'_2, \dots, a'_i, \dots]_p$ be two p -adic numbers represented as two Ruban continued fractions, where $\mathbf{a} = (a_i)_{i \geq 1}$ and $\mathbf{a}' = (a'_i)_{i \geq 1}$ are two sequences not ultimately periodic and satisfying Condition $(*)_\omega$. Assume that $\{-\nu_p(a'_i)\}_{i \geq 1}$ is bounded. If*

$$p > B^{2\omega+2},$$

then either one of α and α' is transcendental or both are in the same quadratic field .

Remark 1. *In the case where $|U_n| = 0$, we obtain Theorem 3.1 in [5]. Applying Theorem 3 with $\mathbf{a} = \mathbf{a}'$ we recover Theorem 3.4 in [5].*

Corollary 1. *Let p be a prime number such that $p \geq 5$. Let $\alpha = [0; a_1, a_2, \dots, a_i, \dots]_p$ and $\alpha' = [0; a'_1, a'_2, \dots, a'_i, \dots]_p$ be two p -adic numbers represented as two Ruban continued fractions, where $\mathbf{a} = (a_i)_{i \geq 1}$ and $\mathbf{a}' = (a'_i)_{i \geq 1}$ are two sequences not ultimately periodic such that $\{-\nu_p(a'_i)\}_{i \geq 1}$ is bounded. Assume that \mathbf{a} and \mathbf{a}' satisfy Condition $(*)_\omega$. If $A = 1$, then either one of α and α' is transcendental or both are in the same quadratic field.*

Corollary 2. *Let p be a prime number such that $p > B^3$. Let $(a_i)_{i \geq 1}$ be a sequence of rational numbers in $\mathbb{Z} \left[\frac{1}{p} \right] \cap (0, p)$ not ultimately periodic such that $\{-\nu_p(a_i)\}_{i \geq 1}$ is bounded. Assume that there exist $(n_k)_{k \geq 1}$, $(\lambda_k)_{k \geq 1}$ and $(r_k)_{k \geq 1}$, three sequences of positive integers where $(r_k)_{k \geq 1}$ is bounded and suppose that the sequence $(a_i)_{i \geq 1}$ satisfies:*

$$a_{n_k+j} = a_{n_k+hr_k+j} \neq p - p^{-1} \quad \text{for } 1 \leq j \leq r_k \quad \text{and } 1 \leq h \leq \lambda_k - 1,$$

and $n_{k+1} > n_k + \lambda_k r_k$ for every $k \geq 1$. If

$$\limsup_{k \rightarrow +\infty} \frac{\lambda_k}{n_k} > 2,$$

then the p -adic number $[0; a_1, a_2, \dots]_p$ is either quadratic or transcendental.

The proof of our theorems are mainly based on the p -adic version of the Schmidt Subspace Theorem, established by Schlickewei [19].

Let $n \geq 2$ be an integer, and let $\mathbf{x} = (x_1, \dots, x_n)$ be an n -tuple of rational numbers. Set $|\mathbf{x}|_\infty = \{\max |x_i|; 1 \leq i \leq n\}$ and $|\mathbf{x}|_p = \{\max |x_i|_p; 1 \leq i \leq n\}$.

Theorem 4. [19] *Let p be a prime number, and let $L_{1,\infty}, \dots, L_{m,\infty}$ be m linearly independent forms in the variable \mathbf{x} and with algebraic real coefficients. Let $L_{1,p}, \dots, L_{m,p}$ be m linearly independent forms with algebraic p -adic coefficients and with the same variable \mathbf{x} . Let $\varepsilon > 0$ be a real number. Then, the set of solutions $\mathbf{x} \in \mathbb{Z}^m$ of the inequality :*

$$\prod_{i=1}^m (|L_{i,\infty}(\mathbf{x})|_\infty |L_{i,p}(\mathbf{x})|_p) \leq |\mathbf{x}|_\infty^{-\varepsilon}$$

is contained in the union of a finite number of proper subspaces of \mathbb{Q}^m .

In addition to this theorem, we need equally certain auxiliary results, which are outlined below.

Lemma 1. [22] *Let $\alpha = [0, a_1, a_2, \dots]_p$ be a p -adic number represented as a Ruban continued fraction with $\frac{p_n}{q_n}$ is its n^{th} convergent. Then, we have:*

$$|q_n|_p = |a_1 \dots a_n|_p, \quad \forall n \geq 1, \quad (3.2)$$

$$\begin{cases} |p_n|_p = |a_0 \dots a_n|_p \quad \forall n \geq 1, \text{ if } a_0 \neq 0 \\ |p_1|_p = 1, \quad |p_n|_p = |a_2 \dots a_n|_p \quad \forall n \geq 2, \text{ if } a_0 = 0, \end{cases} \quad (3.3)$$

$$|q_n|_p < |q_{n+1}|_p \quad \text{and} \quad |p_n|_p < |p_{n+1}|_p. \quad (3.4)$$

Lemma 2. [16] *Let $\alpha = [0, a_1, a_2, \dots]_p$ and $\alpha' = [0, a'_1, a'_2, \dots]_p$ be two p -adic numbers represented as two Ruban continued fractions having the same first $(n+1)$ partial quotients. Then,*

$$|\alpha - \alpha'|_p \leq |q_n|_p^{-2}.$$

Lemma 3. [16] *Let $\alpha = [0, a_1, a_2, \dots]_p$ be a p -adic number represented as a Ruban continued fraction and let $\frac{p_n}{q_n}$ be its n^{th} convergent. Then,*

$$\frac{q_{n-1}}{q_n} = [0, a_n, a_{n-1}, \dots, a_1]_p.$$

Lemma 4. [5] *Let $\alpha = [a_0, a_1, a_2, \dots]_p$ be a p -adic number represented as a Ruban continued fraction such that $\frac{p_n}{q_n}$ is its n^{th} convergent. Let $A = \max\{a_i; i \in \mathbb{N}\}$. Then, we get the following inequalities:*

$$q_n \leq \left(\frac{A + \sqrt{A^2 + 4}}{2} \right)^n, \quad p_n \leq \left(\frac{A + \sqrt{A^2 + 4}}{2} \right)^{n+1}, \quad \text{for all } n \geq 0.$$

4 Proofs of results

Proof of Theorem 2: We suppose that α is a p -adic algebraic number (over \mathbb{Q}). By assumption, there exist a nonzero integer quadruple (a, b, c, d) and an infinite set \mathcal{N} of distinct positive integers such that

$$a s_{n-1} + b r_{n-1} + c s_n + d r_n = 0, \forall n \in \mathcal{N}. \quad (4.1)$$

First, note that $(a, b) \neq (0, 0)$, since otherwise dividing (4.1) by q_n and letting n tend to infinity along \mathcal{N} would imply that α is rational. Similarly, $(c, d) \neq (0, 0)$. Now without loss of generality, we can assume that $a \neq 0$ and so we define three independent linear forms with algebraic real coefficients.

$$L_{i,\infty}(X_1, X_2, X_3) = X_i, \text{ for } 1 \leq i \leq 3,$$

We also consider three independent linear forms with algebraic p -adic coefficients:

$$L_{1,p}(X_1, X_2, X_3) = \left(\alpha \frac{b}{a} + 1\right)X_1 + \alpha \frac{c}{a}X_2 + \alpha \frac{d}{a}X_3,$$

$$L_{2,p}(X_1, X_2, X_3) = \alpha X_2 - X_3,$$

$$L_{3,p}(X_1, X_2, X_3) = X_1,$$

Notice that r_n and s_n for $n \geq 1$ are rational. From this perspective, we set

$M_n = \max(|r_n|_p, |s_n|_p)$ (in our case $M_n = |s_n|_p$), $\mathbf{R}_n = r_n M_n$ and $\mathbf{S}_n = s_n M_n$. So, $M_n, \mathbf{R}_n, \mathbf{S}_n \in \mathbb{N}$. In order to apply Theorem 4, we consider the product of the linear forms given before at the integer points

$$\mathbf{X} = (X_1, X_2, X_3) = (\mathbf{R}_{n-1}, \mathbf{S}_n, \mathbf{R}_n).$$

We obtain from (4.1):

$$|L_{1,p}(\mathbf{X})|_p = |M_n|_p |\alpha s_{n-1} - r_{n-1}|_p.$$

Since $\left(\frac{r_n}{s_n}\right)_{n \geq 1}$ is the n^{th} convergent of α and from Proposition 1, we get:

$$|L_{1p}(\mathbf{X})|_p \leq \frac{|M_n|_p}{|s_{n-1}|_p} \quad (4.2)$$

From Proposition 1 and (4.2), it follows that

$$\prod_{i=1}^3 |L_{i,p}(\mathbf{X})|_p \leq \frac{|M_n|_p^3 |r_{n-1}|_p}{|s_n|_p |s_{n-1}|_p} \leq \frac{|M_n|_p^3}{|s_n|_p}, \quad (4.3)$$

From Lemma 2 we have $|s_n|_p = |a_1 \dots a_n|_p$ and since $|a_k|_p \geq p$ for all $k \geq 1$, then we obtain $|s_n|_p \geq p^n$. Therefore, from (4.3), we get

$$\prod_{i=1}^3 |L_{i,p}(\mathbf{X})|_p \leq \frac{|M_n|_p^3}{p^n}.$$

On the other hand, we have:

$$\prod_{i=1}^3 |L_{i,\infty}(\mathbf{X})|_\infty = |M_n|_\infty^3 |r_{n-1} s_n r_n|_\infty$$

Referring to inequalities recorded in Lemma 4, we get:

$$\prod_{i=1}^3 |L_{i,\infty}(\mathbf{X})|_{\infty} \leq |M_n|_{\infty}^3 B^{3n}$$

This easily implies that for any $\varepsilon > 0$ we have:

$$|\mathbf{X}|_{\infty}^{\varepsilon} \prod_{i=1}^3 |L_{i,\infty}(\mathbf{X})|_{\infty} \leq |M_n^{3+\varepsilon}|_{\infty} B^{(3+\varepsilon)n}.$$

As $\{-\nu_p(a_i)\}_{i \geq 1}$ is bounded, there exists $\delta \in \mathbb{N}^*$ such that $-\nu_p(a_i) \leq \delta$ for all $i \geq 1$. This yields $|M_n|_{\infty} \leq p^{\delta n}$. Therefore, for any $\varepsilon > 0$ we get the following inequality:

$$|\mathbf{X}|_{\infty}^{\varepsilon} \prod_{i=1}^3 (|L_{i,\infty}(\mathbf{X})|_{\infty} |L_{i,p}(\mathbf{X})|_p) \leq \frac{|M_n^{\varepsilon}|_{\infty} B^{(3+\varepsilon)n}}{p^n} \leq \frac{B^{(3+\varepsilon)n}}{p^{(1-\delta\varepsilon)n}}.$$

Choosing $\varepsilon = \frac{1}{\delta^2}$ for δ large enough and based on the fact that $p > B^3$, we get

$$\prod_{i=1}^3 (|L_{i,\infty}(\mathbf{X})|_{\infty} |L_{i,p}(\mathbf{X})|_p) \leq \frac{1}{|\mathbf{X}|_{\infty}^{\varepsilon}}.$$

Thus, according to Theorem 4, the points $(\mathbf{R}_{n-1}, \mathbf{S}_n, \mathbf{R}_n)$ lie in a finite number of proper subspaces of \mathbb{Q}^3 , which implies the existence of a non-zero integer triplet (x_1, x_2, x_3) and an infinite subset \mathcal{N}_1 of \mathcal{N} of distinct positive integers such that

$$x_1 \mathbf{R}_{n-1} + x_2 \mathbf{S}_n + x_3 \mathbf{R}_n = 0, \forall n \in \mathcal{N}_1.$$

From this equation, we have

$$x_1 r_{n-1} + x_2 s_n + x_3 r_n = 0. \quad (4.4)$$

We suppose that $x_1 \neq 0$. At this stage of analysis, combining (4.1) and (4.4) we see that there exists a nonzero element $(y_1, y_2, y_3, y_4) \in \mathbb{Q}^4$ such that

$$r_{n-1} = y_1 s_n + y_2 r_n \quad \text{and} \quad s_{n-1} = y_3 s_n + y_4 r_n, \quad \forall n \in \mathcal{N}_1.$$

Letting n tend to infinity along \mathcal{N}_1 , we obtain

$$\alpha = \frac{y_1 + y_2 \alpha}{y_3 + y_4 \alpha}. \quad (4.5)$$

So α is quadratic.

Proof of Theorem 3: Using the same notation as in the statement of Theorem 3, we let $(U_n)_{n \geq 1}$ and $(V_n)_{n \geq 1}$ denote the sequences satisfying Condition $(*)_{\omega}$. We suppose now that α and α' are two p -adic algebraic numbers of degree > 2 . For $n \geq 1$, set $v_n = |V_n|$ and $u_n = |U_n|$. Let $\left(\frac{p_n}{q_n}\right)_{n \geq 1}$ be the sequence of convergents of α' . By assumption, we have

$$\frac{p_{v_n+u_n}}{q_{v_n+u_n}} = [0; U_n \bar{V}_n]_p,$$

and according to Lemma 3, we get:

$$\frac{q_{v_n+u_n-1}}{q_{v_n+u_n}} = [0; V_n \bar{U}_n]_p.$$

It is clear that α and $\frac{q_{v_n+u_n-1}}{q_{v_n+u_n}}$ have the same first $(v_n + 1)$ partial quotients. Therefore, from Lemma 2, we get:

$$|q_{v_n+u_n}\alpha - q_{v_n+u_n-1}|_p < \frac{|q_{v_n+u_n}|_p}{|q_{v_n}|_p^2}. \quad (4.6)$$

When $v_n \rightarrow +\infty$, we have $|q_{v_n}|_p \rightarrow +\infty$; which implies that:

$$\lim_{n \rightarrow +\infty} \frac{q_{v_n+u_n-1}}{q_{v_n+u_n}} = \alpha \quad (\text{in } \mathbb{Q}_p). \quad (4.7)$$

Since $\left(\frac{p_n}{q_n}\right)_{n \geq 1}$ is the sequence of convergents of α' and from Proposition 1, we get the following inequalities:

$$|q_{v_n+u_n}\alpha' - p_{v_n+u_n}|_p < |q_{v_n+u_n}|_p^{-1} \quad \text{and} \quad |q_{v_n+u_n-1}\alpha' - p_{v_n+u_n-1}|_p < |q_{v_n+u_n-1}|_p^{-1}. \quad (4.8)$$

Let us consider now the following four independent linear forms with algebraic real coefficients :

$$L_{i,\infty}(X_1, X_2, X_3, X_4) = X_i, \text{ for } 1 \leq i \leq 4,$$

Additionally, we consider the following four independent linear forms with algebraic p -adic coefficients:

$$L_{1,p}(X_1, X_2, X_3, X_4) = \alpha X_1 - X_2,$$

$$L_{2,p}(X_1, X_2, X_3, X_4) = \alpha' X_1 - X_3,$$

$$L_{3,p}(X_1, X_2, X_3, X_4) = \alpha' X_2 - X_4,$$

$$L_{4,p}(X_1, X_2, X_3, X_4) = X_4.$$

It is well known that p_n and q_n are rational for each positive integer n . Set

$M_n = \max(|p_n|_p, |q_n|_p)$ (in our case $M_n = |q_n|_p$), $\mathbf{P}_n = p_n M_n$ and $\mathbf{Q}_n = q_n M_n$. Clearly, $M_n, \mathbf{P}_n, \mathbf{Q}_n \in \mathbb{N}$. Set

$$\mathbf{X} = (X_1, X_2, X_3, X_4) = (\mathbf{Q}_{v_n+u_n}, \mathbf{Q}_{v_n+u_n-1}, \mathbf{P}_{v_n+u_n}, \mathbf{P}_{v_n+u_n-1}).$$

From (4.6) and (4.8), we get immediately:

$$\prod_{i=1}^4 |L_{i,p}(\mathbf{X})|_p \leq \frac{|M_{v_n+u_n}|_p^4}{|q_{v_n}|_p^2}, \quad (4.9)$$

From Lemma 2, we have $|q_{v_n}|_p = |a'_1 \dots a'_{v_n}|_p$. As $|a'_k|_p \geq p$ for all $k \geq 1$, then $|q_{v_n}|_p \geq p^{v_n}$. We conclude from (4.9), the following inequality:

$$\prod_{i=1}^4 |L_{i,p}(\mathbf{X})|_p \leq \frac{|M_{v_n+u_n}|_p^4}{p^{2v_n}}.$$

On the other hand, we have

$$\prod_{i=1}^4 |L_{i,\infty}(\mathbf{X})|_\infty = |M_{v_n+u_n}|_\infty^4 |q_{v_n+u_n} q_{v_n+u_n-1} p_{v_n+u_n} p_{v_n+u_n-1}|_\infty.$$

From Lemma 4, we get:

$$\prod_{i=1}^4 |L_{i,\infty}(\mathbf{X})|_\infty \leq |M_{v_n+u_n}|_\infty^4 B^{4(v_n+u_n)}.$$

This easily implies that for any $\varepsilon > 0$ we have:

$$|\mathbf{X}|_\infty^\varepsilon \prod_{i=1}^4 |L_{i,\infty}(\mathbf{X})|_\infty \leq |M_{v_n+u_n}|_\infty^{4+\varepsilon} B^{(v_n+u_n)(4+\varepsilon)}.$$

With respect to the condition $\{-\nu_p(a'_i)\}_{i \geq 1}$ is bounded, we see that it exists $m \in \mathbb{N}^*$ such that $-\nu_p(a'_i) \leq m$ for all $i \geq 1$, which implies that $|M_{v_n+u_n}|_\infty \leq p^{m(v_n+u_n)}$. Finally, we get the following inequality :

$$|\mathbf{X}|_\infty^\varepsilon \prod_{i=1}^4 (|L_{i,\infty}(\mathbf{X})|_\infty |L_{i,p}(\mathbf{X})|_p) \leq \frac{|M_{v_n+u_n}|_\infty^{4+\varepsilon} B^{(v_n+u_n)(4+\varepsilon)}}{p^{2v_n}} \leq \frac{B^{(4+\varepsilon)(v_n+u_n)}}{p^{2v_n - m\varepsilon(v_n+u_n)}}, \quad \forall \varepsilon > 0.$$

The assumption (iv) of Condition $(*)_\omega$ yields that $u_n + v_n \leq \omega' v_n$ with $\omega' = \omega + 1$. Furthermore, we obtain:

$$|\mathbf{X}|_\infty^\varepsilon \prod_{i=1}^4 (|L_{i,\infty}(\mathbf{X})|_\infty |L_{i,p}(\mathbf{X})|_p) \leq \left(\frac{B^{(4+\varepsilon)\omega'}}{p^{2-m\omega'\varepsilon}} \right)^{v_n}.$$

Hence, by taking $\varepsilon = \frac{1}{m^2}$ for m large enough and by using the fact that $p > B^{2\omega+2}$, we get

$$\prod_{i=1}^4 (|L_{i,\infty}(\mathbf{X})|_\infty |L_{i,p}(\mathbf{X})|_p) \leq \frac{1}{|\mathbf{X}|_\infty^\varepsilon}.$$

Applying Theorem 4 we see that the points $(\mathbf{Q}_{v_n+u_n}, \mathbf{Q}_{v_n+u_n-1}, \mathbf{P}_{v_n+u_n}, \mathbf{P}_{v_n+u_n-1})$ lie in a finite number of proper subspaces of \mathbb{Q}^4 . This implies the existence of a non-zero integer quadruple (y_1, y_2, y_3, y_4) and an infinite set \mathcal{N}_2 of distinct positive integers such that

$$y_1 \mathbf{Q}_{v_n+u_n} + y_2 \mathbf{Q}_{v_n+u_n-1} + y_3 \mathbf{P}_{v_n+u_n} + y_4 \mathbf{P}_{v_n+u_n-1} = 0, \quad \forall n \in \mathcal{N}_2.$$

From this equation, we have

$$y_1 q_{v_n+u_n} + y_2 q_{v_n+u_n-1} + y_3 p_{v_n+u_n} + y_4 p_{v_n+u_n-1} = 0, \quad (4.10)$$

From (4.10) and Theorem 2, we conclude that α' is quadratic.

Dividing (4.10) by $q_{v_n+u_n}$ and letting n tend to infinity along \mathcal{N}_2 , we obtain from (4.7) that

$$y_1 + y_2 \alpha + y_3 \alpha' + y_4 \alpha \alpha' = 0. \quad (4.11)$$

Furthermore, referring to (4.11), we get

$$\alpha' = -\frac{y_1 + y_2\alpha}{y_3 + y_4\alpha} \quad (4.12)$$

Since α' is quadratic, we conclude from (4.12) that α is also quadratic.

Proof of Corollary 1. If $A = 1$, then $B = \phi$ with ϕ being the golden ratio. Moreover, if $p \geq 5$, then $p > \phi^3$. Applying Theorem 3 we get statement of corollary.

Proof of Corollary 2. The fact that $(r_k)_{k \geq 1}$ is a bounded sequence implies that there exist $\eta \in \mathbb{N}$ and an infinite set K of positive integers such that:

$$\forall k \in K, \quad r_k = \eta$$

Additionally, using the assumption that $\{-\nu_p(a_i)\}_{i \geq 1}$ is bounded, there exists an infinite set $K' \subset K$ such that

$$a_{n_k+j} = b_j, \quad \forall 1 \leq j \leq \eta \text{ and } \forall k \in K'.$$

Now the result can be obtained by applying Theorem 3 with α being the purely periodic sequence of period $b_\eta \dots b_1$ and $\alpha' = [0; a_1, a_2, \dots]_p$.

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