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On generalized Gauss maps of minimal surfaces sharing hypersurfaces in a projective variety

by

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Abstract

In this article, we study the uniqueness problem for the generalized Gauss maps of minimal surfaces (with the same base) immersed in $\mathbb{R}^{n+1}$ which have the same inverse image of some hypersurfaces in a projective subvariety $V \subset \mathbb{P}^n(\mathbb{C})$. As we know, this is the first time the unicity of generalized Gauss maps on minimal surfaces sharing hypersurfaces in a projective varieties is studied. Our results generalize and improve the previous results in this field.

Key Words: Gauss map, value distribution, holomorphic curve, uniqueness, algebraic dependence, hypersurface.
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1 Introduction and main results

Let $x_1 : S_1 \to \mathbb{R}^{n+1}$ and $x_2 : S_2 \to \mathbb{R}^{n+1}$ be two oriented non-flat minimal surfaces immersed in $\mathbb{R}^{n+1}$ and let $G_1 : S_1 \to \mathbb{P}^n(\mathbb{C})$ and $G_2 : S_2 \to \mathbb{P}^n(\mathbb{C})$ be their generalized Gauss maps. Assume that there is a conformal diffeomorphism Φ of S_1 onto S_2 and the Gauss map of the minimal surface $x_2 \circ \Phi : S_1 \to \mathbb{P}^n(\mathbb{C})$ is given by $G_2 \circ \Phi$. Then $f^1 = G_1$, $f^2 = G_2 \circ \Phi$ are two nonconstant holomorphic maps from S_1 into $\mathbb{P}^n(\mathbb{C})$. In 1993, Fujimoto obtained the following result.

Theorem A (cf. [4, Theorem 1.2]). *Under the notation be as above, let $H_1, \ldots, H_q$ be q hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that*

(a) $(f^1)^{-1}(H_j) = (f^2)^{-1}(H_j)$ for every j,

(b) $f^1 = f^2$ on $\bigcup_{j=1}^q (f^1)^{-1}(H_j) \setminus K$ for a compact subset K of S_1.

Then we have necessarily $f^1 = f^2$

(1) if $q > (n+1)^2 + \frac{n(n+1)}{2}$ for the case where S_1 is complete and has infinite total curvature or

(2) if $q \geq (n+1)^2 + \frac{n(n+1)}{2}$ for the case where $K = \emptyset$ and S_1 and S_2 are both complete and have finite total curvature.

In 2017, J. Park and M. Ru [8] considered the case where f^1 and f^2 are linearly nondegenerate with an addition assumption that $\bigcap_{j=1}^{k}(f^1)^{-1}(H_{i_j}) = \emptyset$ for every $1 \leq i_1 < \cdots < i_k \leq q$ $(k \geq 2)$.

Recently, in [11], the author initially studied the modified defect relation for the Gauss map of a minimal surface into a projective variety with hypersurfaces in subgeneral position. Motivated by the methods of [10, 11], in this paper, we will generalize the above mentioned results to the cases where Gauss maps into a projective subvariety of $\mathbb{P}^n(\mathbb{C})$ have the same inverse image for some hypersurfaces in subgeneral position.

In order to state our results, we recall the following. Let S be an open complete Riemann surface in $\mathbb{R}^{n+1}$. Let f be a holomorphic map from S into an ℓ-dimension projective subvariety V of $\mathbb{P}^n(\mathbb{C})$ and let Q be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. By $\nu_{Q(f)}$ we denote the pull-back of the divisor Q by f. Let $F = (f_0, \ldots, f_n)$ be a reduced representation of f. Assume that, the hypersurface Q has a defining polynomial, denoted again by the same notation Q (throughout this paper) if there is no confusion, given by

$$Q(x_0, \ldots, x_n) = \sum_{I \in \mathcal{T}_d} a_I x^I,$$

where $\mathcal{T}_d = \{(i_0, \ldots, i_n) \in \mathbb{Z}_+^{n+1}; i_0 + \cdots + i_n = d\}$, $a_I \in \mathbb{C}$ are not all zero for $I \in \mathcal{T}_d$ and $x^I = x_0^{i_0} \ldots x_n^{i_n}$ for each $i = (i_0, \ldots, i_n)$. We set

$$Q(F) = \sum_{I \in \mathcal{T}_d} a_I f^I,$$

where $f^I = f_0^{i_0} \ldots f_n^{i_n}$ for each $I \in \mathcal{T}_d$. Throughout this paper, for each given hypersurface Q we assume that $\|Q\| = (\sum_{I \in \mathcal{T}_d} |a_I|^2)^{1/2} = 1$.

Denote by $I(V)$ the ideal of homogeneous polynomials in $\mathbb{C}[x_0, ..., x_n]$ defining V and by $\mathbb{C}[x_0, ..., x_n]_d$ the vector space of all homogeneous polynomials in $\mathbb{C}[x_0, ..., x_n]$ of degree d including the zero polynomial. Define

$$I_d(V) := \frac{\mathbb{C}[x_0, ..., x_n]_d}{I(V) \cap \mathbb{C}[x_0, ..., x_n]_d} \text{ and } H_V(d) := \dim I_d(V).$$

Denote by $[D]$ the equivalent class in $I_d(V)$ of the element $D \in \mathbb{C}[x_0, ..., x_n]_d$.

For the variety V of $\mathbb{P}^n(\mathbb{C})$ such that $f(S) \subset V$, we say that f is nondegenerate over $I_d(V)$ if there is no $[Q] \in I_d(V) \setminus \{0\}$ such that $Q(F) \equiv 0$.

Let $Q_1, ..., Q_q$ $(q \geq N + 1)$ be q hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. The hypersurfaces $Q_1, ..., Q_q$ are said to be in N-subgeneral position with respect to V if

$$V \cap \left(\bigcap_{j=1}^{N+1} Q_{i_j} \right) = \varnothing \ \forall\ 1 \leq i_1 < \cdots < i_{N+1} \leq q.$$

Our first main result is stated as follows.

Theorem 1.1. *Let V be an ℓ-dimension projective subvariety of $\mathbb{P}^n(\mathbb{C})$. Let S_1, S_2 be non-flat minimal surfaces immersed in $\mathbb{R}^{n+1}$ with the Gauss maps G_1, G_2 into V, respectively. Assume that there is a conformal diffeomorphism Φ of S_1 onto S_2. Let $f^1 = G_1, f^2 = G_2 \circ \Phi$. Let $Q_1, \ldots, Q_q$ be q hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in $N-$subgeneral position with respect to V, $d = \operatorname{lcm}(\deg Q_1, \ldots, \deg Q_q)$ and let k be a positive integer such that:*

(a) $(f^1)^{-1}(Q_j) = (f^2)^{-1}(Q_j)$ *for every* $j \in \{1, \dots, q\}$,

(b) $\bigcap_{j=0}^k (f^1)^{-1}(Q_{i_j}) = \emptyset$ *for every* $1 \le i_0 < \cdots < i_k \le q$,

(c) $f^1 = f^2$ *on* $\bigcup_{j=1}^q (f^1)^{-1}(Q_j)$.

Suppose that f^1 *is nondegenerate over* $I_d(V)$. *If* S^1 *is complete and*

$$q > \frac{2N - \ell + 1}{\ell + 1}\left(M + 1 + \frac{2(\sigma_M - \sigma_{M-\min\{k,\ell\}})}{d} + \frac{M(M+1)}{2d}\right)$$

where $M = H_d(V) - 1$, $\sigma_p = \frac{p(p+1)}{2}$ *for every* $p \ge 0$ *and* $\sigma_p = 0$ *for every* $p \le 0$, *then* $f^1 \equiv f^2$.

Remark 1: If V is the smallest linear subspace of $\mathbb{P}^n(\mathbb{C})$ containing $f^1(S)$ and $Q_1, \dots, Q_q$ are hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position, then $V = \mathbb{P}^\ell(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$, $d = 1$, $N = n, M = \ell$. Therefore, from Theorem 1.1, $f^1 = f^2$ if

$$q > \frac{2n - \ell + 1}{\ell + 1}\left(\ell + 1 + \frac{3\ell(\ell+1)}{2}\right) = \frac{(2n - \ell + 1)(3\ell + 2)}{2}.$$

This condition is always fulfilled if $q > \frac{(n+1)(3n+2)}{2} = (n+1)^2 + \frac{n(n+1)}{2}$ (without any condition on $f^1(S)$). Then this theorem gives an improvement for Theorem A(1).

Theorem 1.2. *Let* V *be an* $\ell-$*dimension projective subvariety of* $\mathbb{P}^n(\mathbb{C})$. *Let* S_1, S_2 *be non-flat minimal surfaces in* $\mathbb{R}^{n+1}$ *with the Gauss maps* G_1, G_2 *into* V, *respectively. Assume that there is a conformal diffeomorphism* Φ *of* S_1 *onto* S_2. *Let* $f^1 = G_1, f^2 = G_2 \circ \Phi$. *Let* $Q_1, \dots, Q_q$ *be* q *hypersurfaces (not containing* V*) of* $\mathbb{P}^n(\mathbb{C})$ *in* $N-$*subgeneral position with respect to* V, $d = \operatorname{lcm}(\deg Q_1, \dots, \deg Q_q)$ *and let* k *be a positive integer such that:*

(a) $(f^1)^{-1}(Q_j) = (f^2)^{-1}(Q_j)$ *for every* $j \in \{1, \dots, q\}$,

(b) $\bigcap_{j=0}^k (f^1)^{-1}(Q_{i_j}) = \emptyset$ *for every* $1 \le i_0 < \cdots < i_k \le q$,

(c) $f^1 = f^2$ *on* $\bigcup_{j=1}^q (f^1)^{-1}(Q_j)$.

If f^1 *is nondegenerate over* $I_d(V)$, S^1 *is complete,* $q \ge 2Mk + 2k$ *and*

$$q > \frac{2N - \ell + 1}{\ell + 1}\left(M + 1 + \frac{2Mkq}{(q + 2(M-1)k)d} + \frac{M(M+1)}{2d}\right)$$

then there is $\left[\frac{q}{2}\right]$ *indices* $i_1, \dots, i_{[q/2]} \in \{1, \dots, q\}$ *such that*

$$\frac{Q_{i_1}(F^1)}{Q_{i_1}(F^2)} = \cdots = \frac{Q_{i_{[q/2]}}(F^1)}{Q_{i_{[q/2]}}(F^2)}$$

for any two representations F^1, F^2 *of* f^1, f^2, *respectively.*

Remark 2: In the above theorem, suppose that $V = \mathbb{P}^n(\mathbb{C})$, $Q_1, \dots, Q_q$ are hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position. Then $d = 1$, $M = N = \ell = n$. Therefore, from the above theorem, $f^1 = f^2$ if $q \ge 2nk + 2k$ and

$$q > n + 1 + \frac{2nkq}{q + 2nk - 2k} + \frac{n(n+1)}{2}.$$

Therefore, this result implies the previous result of J. Park and M. Ru in [8].

2 Main lemmas

Let V be ℓ-dimension subvariety of $\mathbb{P}^n(\mathbb{C})$. Let d be a positive integer. Throughout this section and Section 3, we fix a $\mathbb{C}$-ordered basis $\mathcal{V} = ([v_0], \dots, [v_M])$ of $I_d(V)$, where $v_i \in H_d$ and $M = H_V(d) - 1$.

Let S be an open Riemann surface and let z is a conformal coordinate. Let f be a holomorphic map of S into V, which is nondegenerate over $I_d(V)$. Suppose that $F = (f_0, \dots, f_n)$ is a reduced representation of f. We set

$$F = (v_0(F), \dots, v_M(F))$$

and

$$F_p := F^{(0)} \wedge F^{(1)} \wedge \cdots \wedge F^{(p)} : S \to \bigwedge_{p+1} \mathbb{C}^{M+1}$$

for $0 \le p \le M$, where

- $F^{(0)} := F = (v_0(F), \dots, v_M(F))$,
- $F^{(l)} = F^{(l)} := \left(v_0(F)^{(l)}, \dots, v_M(F)^{(l)}\right)$ for each $l = 0, 1, \dots, p$,
- $v_i(F)^{(l)}$ $(i = 0, \dots, M)$ is the l^{th}- derivatives of $v_i(F)$ taken with respect to z.

The norm of F_p is given by

$$|F_p| := \left(\sum_{0 \le i_0 < i_1 < \cdots < i_p \le M} \left| W(v_{i_0}(F), \dots, v_{i_p}(F)) \right|^2 \right)^{1/2},$$

where

$$W(v_{i_0}(F), \dots, v_{i_p}(F)) := \det \left(v_{i_j}(F)^{(l)} \right)_{0 \le l, j \le p}.$$

Denote by $\langle , \rangle$ the canonical hermitian product on $\bigwedge^{k+1} \mathbb{C}^{M+1}$ $(0 \le k \le M)$. For two vectors $A \in \bigwedge^{k+1} \mathbb{C}^{M+1}$ $(0 \le k \le M)$ and $B \in \bigwedge^{p+1} \mathbb{C}^{M+1}$ $(0 \le p \le k)$, there is one and only one vector $C \in \bigwedge^{k-p} \mathbb{C}^{M+1}$ satisfying

$$\langle C, D \rangle = \langle A, B \wedge D \rangle \ \forall D \in \bigwedge^{k-p} \mathbb{C}^{M+1}.$$

The vector C is called the interior product of A and B, and denoted by $A \vee B$.

Now, for a hypersurface Q of degree d in $\mathbb{P}^n(\mathbb{C})$, we have

$$[Q] = \sum_{i=0}^{M} a_i [v_i].$$

Hence, we associate Q with the vector $(a_0, \dots, a_M) \in \mathbb{C}^{M+1}$ and define $F_p(Q) = F_p \vee H$. Then, we may see that

$$F_0(Q) = a_0 v_0(F) + \cdots + a_M v_M(F) = Q(F),$$

$$|F_p(Q)| = \left(\sum_{0 \le i_1 < \cdots < i_p \le M} \sum_{l \ne i_1, \dots, i_p} a_l \left| W(v_l(F), v_{i_1}(F), \dots, v_{i_p}(F)) \right|^2 \right)^{1/2}.$$

For $0 \le p \le M$, the p^{th}-contact function of f for Q is defined by

$$\varphi_p(Q) := \frac{|F_p(Q)|^2}{|F_p|^2}.$$

Lemma 2.1 (cf. [9, Lemma 3]). *Let $Q_1, ..., Q_q$ be q $(q > 2N - \ell + 1)$ hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in N-subgeneral position with respect to V of the same degree d. Then, there are positive rational constants ω_i $(1 \le i \le q)$ satisfying the following:*

i) $0 < \omega_i \le 1$ $\forall i \in \{1, ..., q\}$,

ii) Setting $\tilde{\omega} = \max_{j \in Q} \omega_j$, one gets $\sum_{j=1}^{q} \omega_j = \tilde{\omega}(q - 2N + \ell - 1) + \ell + 1$.

iii) $\dfrac{\ell + 1}{2N - \ell + 1} \le \tilde{\omega} \le \dfrac{\ell}{N}$.

iv) For each $R \subset \{1, ..., q\}$ with $\sharp R = N + 1$, then $\sum_{i \in R} \omega_i \le \ell + 1$.

v) Let $E_i \ge 1$ $(1 \le i \le q)$ be arbitrarily given numbers. For each $R \subset \{1, ..., q\}$ with $\sharp R = N + 1$, there is a subset $R^o \subset R$ such that $\sharp R^o = \operatorname{rank}\{[Q_i]\}_{i \in R^o} = \ell + 1$ and

$$\prod_{i \in R} E_i^{\omega_i} \le \prod_{i \in R^o} E_i.$$

The following theorem is due to the author in recent works [11, 12, 13].

Theorem 2.2 (cf. [11, Theorem 3.3],[12, Theorem 3.5],[13, Theorem 2.7]). *Let the notations be as above and let $\tilde{\omega}$ be the constant defined in the Lemma 2.1 with respect to the hypersurfaces $Q_1, \dots, Q_q$. Then, for every $\epsilon > 0$, there exist a positive number δ (> 1) and C, depending only on ϵ and Q_j such that*

$$\begin{aligned} &\mathrm{dd^c} \log \frac{\prod_{p=0}^{M-1} |F_p|^{2\epsilon}}{\prod_{1 \le j \le q, 0 \le p \le M-1} \log^{2\omega_j} (\delta / \varphi_p(Q_j))} \\ &\ge C \left(\frac{|F_0|^{2(\tilde{\omega}(q - (2N - k + 1)) - M + k)} |F_M|^2}{\prod_{j=1}^{q} (|F_0(Q_j)|^2 \prod_{p=0}^{M-1} \log^2(\delta / \varphi_p(Q_j)))^{\omega_j}} \right)^{\frac{2}{M(M+1)}} \mathrm{dd^c}|z|^2. \end{aligned}$$

Theorem 2.3 (cf. [5, Proposition 2.5.7]). *Set $\tau_m = \sum_{p=1}^{m} \sigma_m$ for each integer m. We have*

$$\mathrm{dd^c} \log(|F_0|^2 \cdots |F_{M-1}|^2) \ge \frac{\tau_M}{\sigma_M} \left(\frac{|F_0|^2 \cdots |F_M|^2}{|F_0|^{2\sigma_{M+1}}} \right)^{1/\tau_M} \mathrm{dd^c}|z|^2.$$

Theorem 2.4. *Let the notations be as above and let the assumption be as in Lemma 2.1, we have*

$$\nu_{F_M^1} \ge \sum_{j=1}^{q} \omega_j \nu_{Q_j(F)} - (\sigma_M - \sigma_{M - \min\{k, \ell\}}) \nu^{[1]}_{\prod_{j=1}^{q} Q_j(F)}.$$

Proof. For a point $a \in \bigcup_{j=1}^{q} (f^1)^{-1}(Q_j)$, since $\{Q_j\}_{j=1}^{q}$ is in N-subgeneral position with respect to V, there are at most N indices j such that $Q_j(F^1)(a) = 0$. Then, there is a subset $R \subset \{1, \dots, q\}$ with $\sharp R = N + 1$ such that $Q_j(F^1)(a) \ne 0$ $\forall j \notin R$. Applying Lemma 2.1, there exists a subset $R^0 \subset R$ with $\sharp R^o = \ell + 1$ such that $\operatorname{rank}_{\mathbb{C}}\{[Q_j]; j \in R^o\} = \ell + 1$ and

$$\sum_{j=1}^{q} \omega_j \nu_{Q_j(F)}(a) = \sum_{j \in R} \omega_j \nu_{Q_j(F^1)}(a) \le \sum_{j \in R^o} \nu_{Q_j(F^1)}(a).$$

We set $k' = \min\{k, \ell\}$. Since there are at most k' indices $j \in R^o$ such that $Q_j(F^1)(a) = 0$, we also may assume further that $R^o = \{1, \dots, \ell + 1\}$, $Q_j(F^1)(a) \neq 0$ for all $j > k', j \in R^o$. By the basis property of the wronskian, we have

$$\nu_{F^1_M}(a) \geq \min_{\alpha} \left\{ \sum_{j=1}^{k'} \max\{0, \nu_{Q_j(f^1)}(a) - (M - \alpha(j))\} \right\} \geq \sum_{j=1}^{k'} \nu_{Q_j(f^1)}(a) - (\sigma_M - \sigma_{M-k'}),$$

where the minimum is taken over all bijections $\alpha : \{1, \dots, k'\} \to \{0, \dots, k' - 1\}$. Thus

$$\nu_{F^1_M} \geq \sum_{j=1}^{q} \omega_j \nu_{Q_j(F)} - (\sigma_M - \sigma_{M-\min\{k,\ell\}}) \nu^{[1]}_{\prod_{j=1}^q Q_j(F)}.$$

The theorem is proved. □

Lemma 2.5 (Generalized Schwarz's Lemma [1]). *Let v be a non-negative real-valued continuous subharmonic function on $\Delta(R) = \{z \in \mathbb{C}; |z| < R\}$. If v satisfies the inequality $\Delta \log v \geq v^2$ in the sense of distribution, then*

$$v(z) \leq \frac{2R}{R^2 - |z|^2}.$$

3 Holomorphic curves from complex discs into projective varieties

Lemma 3.1. *Let V be an ℓ-dimension projective subvariety of $\mathbb{P}^n(\mathbb{C})$. Let $Q_1, \dots, Q_q$ be q hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in $N-$subgeneral position with respect to V and let d be the least common multiple of $\deg Q_1, \dots, \deg Q_q$. Let $f^1, \dots, f^m$ be m holomorphic maps from $\Delta(R)$ into V $(1 \leq m \leq n + 1)$, which are nondegenerate over $I_d(V)$. Assume that there exists a holomorphic function h on $\Delta(R)$ satisfying*

$$\lambda \nu_h + \sum_{i=1}^{m} \nu_{F^i_M} \geq \sum_{i=1}^{m} \sum_{j=1}^{q} \omega_j \nu_{Q_j(F^i)} \text{ and } |h| \leq \prod_{i=1}^{m} \|F^i\|^{\rho},$$

where $F^i = (F^i_0, \dots, F^i_n)$ is a reduced representation of f^i $(1 \leq i \leq m)$, λ and ρ are non-negative numbers. Then for an arbitrarily given ϵ satisfying

$$\gamma = \sum_{j=1}^{q} \omega_j - M - 1 - \frac{\lambda\rho}{d} > \epsilon \left(\sigma_{M+1} + \frac{\rho}{d} \right).$$

the pseudo-metric $d\tau^2 = \eta^{2/m} |dz|^2$, where

$$\eta = \left(|h|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|F^i_0|^{\gamma - \epsilon(\sigma_{M+1} + \frac{\rho}{d})} |F^i_M| \prod_{p=0}^{M} |F^i_p|^{\epsilon}}{\prod_{j=1}^{q} \left(|Q_j(F^i)| \cdot \prod_{p=1}^{M-1} \log(\delta^i / \varphi^i_p(Q_j)) \right)^{\omega_j}} \right)^{\frac{1}{\sigma_M + \epsilon \tau_M}}$$

and δ^i is the number satisfying the conclusion of Theorem 2.2 with respect to the map f^i, is continuous and has strictly negative curvature.

Here and throughout this paper, F_p^i and φ_p^i are defined with respect to the map f^i. For simplicity, we sometimes write $\prod_{i,j}$ and $\prod_{j,p}$ for $\prod_{i=1}^m\prod_{j=1}^q$ and $\prod_{j=1}^q\prod_{p=1}^{M-1}$, respectively.

Proof. We see that the function η is continuous at every point z with $\prod_{i,j}Q_j(F^i)(z)\neq 0$. For a point $z_0\in\Delta(R)$ such that $\prod_{i,j}Q_j(F^i)(z_0)=0$, we have

$$\nu_\eta(z_0)\geq\frac{1}{\sigma_M+\epsilon\tau_M}\left(\lambda\nu_h(z_0)+\sum_{i=1}^m\nu_{F_M^i}(z_0)-\sum_{i=1}^m\sum_{j=1}^q\omega_j\nu_{Q_j(F^i)}(z_0)\right)\geq 0.$$

This implies that $d\tau^2$ is a continuous pseudo-metric on $\Delta(R)$.

We now prove that $d\tau^2$ has strictly negative curvature on $\Delta(R)$. Again, we have

$$\sum_{i=1}^m\mathrm{dd^c}\log\frac{|F_M^i|^{1+\epsilon}}{\prod_{j=1}^q|Q_j(F)|^{\omega_j}}+(\lambda+\epsilon)\mathrm{dd^c}\log|h|\geq 0.$$

Let Ω be the Fubini-Study form of $\mathbb{P}^n(\mathbb{C})$ and denote by Ω_{f^i} the pull-back of Ω by the map f^i $(1\leq i\leq m)$. By Theorems 2.2 and 2.3, we have

$$\begin{aligned}
\mathrm{dd^c}\log\eta^{1/m}&\geq\frac{\gamma-\epsilon(\sigma_{M+1}+\frac{\rho}{d})}{m(\sigma_M+\epsilon\tau_M)}d\sum_{i=1}^m\Omega_{f^i}\\
&+\frac{\epsilon}{4m(\sigma_M+\epsilon\tau_M)}\sum_{i=1}^m\mathrm{dd^c}\log\left(|F_0^i|^2\cdots|F_{M-1}^i|^2\right)\\
&+\frac{1}{2m(\sigma_M+\epsilon\tau_M)}\sum_{i=1}^m\mathrm{dd^c}\log\frac{\prod_{p=0}^{M-1}|F_p^i|^{2(\frac{\epsilon}{2})}}{\prod_{p=0}^{M-1}\log^{2\omega_j}(\delta^i/\varphi_p^i(Q_j))}\\
&\geq\frac{\epsilon\tau_M}{4m\sigma_M(\sigma_M+\epsilon\tau_M)}\sum_{i=1}^m\left(\frac{|F_0^i|^2\cdots|F_M^i|^2}{|F_0^i|^{2\sigma_{M+1}}}\right)^{\frac{1}{\tau_M}}\mathrm{dd^c}|z|^2\\
&+C_0\sum_{i=1}^m\left(\frac{|F_0^i|^{2(\tilde{\omega}(q-2N+k-1)-M+k)}|F_M^i|^2}{\prod_{j=1}^q(|Q_j(F^i)|^2\prod_{p=0}^{M-1}\log^2(\delta^i/\varphi_p^i(Q_j)))^{\omega_j}}\right)^{\frac{1}{\sigma_M}}\mathrm{dd^c}|z|^2 \qquad (1)\\
&\geq\frac{\epsilon\tau_M}{4\sigma_M(\sigma_M+\epsilon\tau_M)}\left(\prod_{i=1}^m\frac{|F_0^i|^2\cdots|F_M^i|^2}{|F_0^i|^{2\sigma_{M+1}}}\right)^{\frac{1}{m\tau_M}}\mathrm{dd^c}|z|^2\\
&+mC_0\left(\prod_{i=1}^m\frac{|F_0^i|^{2(\tilde{\omega}(q-2N+k-1)-M+k)}|F_M^i|^2}{\prod_{j=1}^q(|Q_j(F^i)|^2\prod_{p=0}^{M-1}\log^2(\delta^i/\varphi_p^i(Q_j)))^{\omega_j}}\right)^{\frac{1}{m\sigma_M}}\mathrm{dd^c}|z|^2\\
&\geq C_1\left(\prod_{i=1}^m\frac{|F_0^i|^{\tilde{\omega}(q-2N+k-1)-M+k-\epsilon\sigma_{M+1}}|F_M^i|\prod_{p=0}^M|F_p^i|^\epsilon}{\prod_{j=1}^q\left(|Q_j(F^i)|\prod_{p=0}^{M-1}\log(\delta^i/\varphi_p^i(Q_j))\right)^{\omega_j}}\right)^{\frac{2}{m(\sigma_M+\epsilon\tau_M)}}\mathrm{dd^c}|z|^2
\end{aligned}$$

for some positive constants C_0,C_1, where the last inequality comes from Hölder's inequality. On the other hand, we have $|h|\leq\prod_{i=1}^m\|F^i\|^\rho\leq\prod_{i=1}^m|F_0^i|^{\frac{\rho}{d}}$ and

$$\prod_{i=1}^m|F_0^i|^{\tilde{\omega}(q-2N+k-1)-M+k-\epsilon\sigma_{M+1}}\geq|h|^{\lambda+\epsilon}\prod_{i=1}^m|F_0^i|^{\gamma-\epsilon(\sigma_{M+1}+\frac{\rho}{d})}.$$

This implies that $\Delta \log \eta^{2/m} \geq C_2 \eta^{2/m}$ for some positive constant C_2. Therefore, $d\tau^2$ has strictly negative curvature. ∎

Lemma 3.2. *Let the notations and the assumption be as in Lemma 3.1. Then for an arbitrarily given ϵ satisfying*

$$\gamma = \sum_{j=1}^{q} \omega_j - M - 1 - \frac{\lambda\rho}{d} > \epsilon(\sigma_{M+1} + \frac{\rho}{d}),$$

there exists a positive constant C, depending only on ϵ, Q_j $(1 \leq j \leq q)$, such that

$$\left(|h|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|F_0^i|^{\gamma-\epsilon(\sigma_{M+1}+\frac{\rho}{d})}|F_M^i|^{1+\epsilon}\prod_{j,p}|F_p^i(Q_j)|^{\epsilon/q}}{\prod_{j=1}^{q}|Q_j(F^i)|^{\omega_j}}\right)^{1/m} \leq C\left(\frac{2R}{R^2-|z|^2}\right)^{\sigma_M+\epsilon\tau_M}.$$

Proof. As in the proof of Lemma 3.1, we have

$$\mathrm{dd^c}\log\eta^{1/m} \leq C_2\eta^{2/m}\mathrm{dd^c}|z|^2.$$

According to Lemma 2.5, this implies that

$$\eta^{1/m} \leq C_3\frac{2R}{R^2-|z|^2},$$

for some positive constant C_3. Then we have

$$\left(|h|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|F_0^i|^{\gamma-\epsilon(\sigma_{M+1}+\frac{\rho}{d})}|F_M^i|\prod_{p=0}^{M}|F_p^i|^{\epsilon}}{\prod_{j=1}^{q}\left(|Q_j(F^i)|\cdot\prod_{p=1}^{M-1}\log(\delta^i/\varphi_p^i(Q_j))\right)^{\omega_j}}\right)^{\frac{1}{m(\sigma_M+\epsilon\tau_M)}} \leq C_3\frac{2R}{R^2-|z|^2}.$$

It follows that

$$\left(|h|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|F_0^i|^{\gamma-\epsilon(\sigma_{M+1}+\frac{\rho}{d})}|F_M^i|^{1+\epsilon}\prod_{j,p}^{i}|F_p^i(Q_j)|^{\frac{\epsilon}{q}}}{\prod_{j=1}^{q}\left(|Q_j(F^i)|\prod_{p=0}^{M-1}(\varphi_p^i(Q_j))^{\frac{\epsilon}{2q}}\log(\delta^i/\varphi_p^i(Q_j))\right)^{\omega_j}}\right)^{\frac{1}{m(\sigma_M+\epsilon\tau_M)}} \leq C_3\frac{2R}{R^2-|z|^2}.$$

Note that the function $x^{\frac{\epsilon}{q}}\log^{\omega}\left(\frac{\delta}{x^2}\right)$ $(\omega > 0, 0 < x \leq 1)$ is bounded. Then we have

$$\left(|h|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|F_0^i|^{\gamma-\epsilon(\sigma_{M+1}+\frac{\rho}{d})}|F_M^i|^{1+\epsilon}\prod_{j,p}|F_p^i(Q_j)|^{\epsilon/q}}{\prod_{j=1}^{q}|Q_j(F^i)|^{\omega_j}}\right)^{\frac{1}{m(\sigma_M+\epsilon\tau_M)}} \leq C_4\frac{2R}{R^2-|z|^2},$$

for a positive constant C_4. The lemma is proved. ∎

Lemma 3.3 (cf. [5, Lemma 1.6.7])**.** *Let $d\sigma^2$ be a conformal flat metric on an open Riemann surface S. Then for every point $p \in S$, there is a holomorphic and locally biholomorphic map Φ of a disk $\Delta(R_0)$ onto an open neighborhood of p with $\Phi(0) = p$ such that Φ is a local isometric, namely the pull-back $\Phi^*(d\sigma^2)$ is equal to the standard (flat) metric on $\Delta(R_0)$, and for some point a_0 with $|a_0| = 1$, the curve $\Phi(\overline{0, R_0a_0})$ is divergent in S (i.e., for any compact set $K \subset S$, there exists an $s_0 < R_0$ such that $\Phi(\overline{0, s_0a_0})$ does not intersect K).*

Theorem 3.4. *Let S be an open Riemann surface and V be an ℓ-dimension projective subvariety of $\mathbb{P}^n(\mathbb{C})$. Let $f^1,\dots,f^m$ be m holomorphic curves from S into V $(1\le m\le n)$. Let $Q_1,\dots,Q_q$ be q hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in $N-$subgeneral position with respect to V and $d=\mathrm{lcm}(\deg Q_1,\dots,\deg Q_q)$. Assume that each f_i is nondegenerate over $I_d(V)$, there exists a holomorphic function h on S satisfying*

$$\lambda\nu_h+\sum_{i=1}^m\nu_{F^i_M}\ge\sum_{i=1}^m\sum_{j=1}^q\omega_j\nu_{Q_j(F^i)}\ \text{and}\ |h|\le\prod_{i=1}^m\|F^i\|^\rho,$$

where $F^i=(F^i_0,\dots,F^i_n)$ is a reduced representation of f^i $(1\le i\le m)$ and the metric

$$ds^2=2|\xi|^{2/m}\cdot\left(\prod_{i=1}^m\|F^i\|\right)^{2/m}|dz^2|,$$

where ξ is a nowhere zero holomorphic function, is complete on S. Then we have

$$q\le\frac{2N-\ell+1}{\ell+1}\left(M+1+\frac{\lambda\rho}{d}+\frac{M(M+1)}{2d}\right).$$

Proof. If there are some hypersurfaces Q_j such that $V\subset Q_j$, for instance they all are $Q_{q-r+1},\dots,Q_q$ $(0\le r\le N-\ell+1)$, then by setting $N'=N-r, q'=q-r$ we have

$$\begin{aligned}&\frac{2N-\ell+1}{\ell+1}\left(M+1+\frac{\lambda\rho}{d}+\frac{M(M+1)}{2d}\right)-q\\&\quad\ge\frac{2N'-\ell+1}{\ell+1}\left(M+1+\frac{\lambda\rho}{d}+\frac{M(M+1)}{2d}\right)-q'\end{aligned}$$

and $Q_1,\dots,Q_{q-r}$ are in N'-subgeneral position with respect to V. Then, without loss of generality, we may assume that $V\not\subset Q_j$ for all $j=1,\dots,q$.

We fix a $\mathbb{C}$-ordered basis $\mathcal{V}=([v_0],\dots,[v_M])$ of $I_d(V)$ as in the Section 3. By replacing Q_i with $Q_i^{d/\deg Q_i}$ $(1\le i\le q)$ if necessary, we may assume that all Q_i $(1\le i\le q)$ are of the same degree d. Suppose that

$$[Q_j]=\sum_{i=0}^M a_{ji}[v_i],$$

where $\sum_{i=0}^M|a_{ji}|^2=1$.

Since f^i $(1\le i\le m)$ is nondegenerate over $I_d(V)$, the contact functions satisfy

$$F^i_p(Q_j)\not\equiv0,\forall 1\le j\le q, 0\le p\le M.$$

Then, for each j,p $(1\le j\le q, 0\le p\le M)$, we may choose $i_1,\dots,i_p$ with $0\le i_1<\dots<i_p\le M$ such that

$$\psi(F^i)_{jp}=\sum_{s\ne i_1,\dots,i_p}a_{js}W(v_s(F^i),v_{i_1}(F^i),\dots,v_{i_p}(F^i))\not\equiv0.$$

We note that $\psi(F^i)_{j0} = F^i_0(Q_j) = Q_j(F^i)$ and $\psi(F^i)_{jM} = F^i_M$.

Suppose contrarily that

$$q > \frac{2N-\ell+1}{\ell+1}\left(M+1+\frac{\lambda\rho}{d}+\frac{M(M+1)}{2d}\right).$$

From Theorem 2.1, we have

$$(q-2N+\ell-1)\tilde{\omega} = \sum_{j=1}^{q}\omega_j - \ell - 1;\ \tilde{\omega} \geq \omega_j > 0 \text{ and } \tilde{\omega} \geq \frac{\ell+1}{2N-\ell+1}.$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{q}\omega_j - M - 1 - \frac{\lambda\rho}{d} &\geq \tilde{\omega}(q-2N+\ell-1) - M + \ell - \frac{\lambda\rho}{d}\\ &\geq \frac{\ell+1}{2N-\ell+1}(q-2N+\ell-1) - M + \ell - \frac{\lambda\rho}{d}\\ &= \frac{\ell+1}{2N-\ell+1}\left(q - \frac{(2N+\ell-1)(M+1+\frac{\lambda\rho}{d})}{\ell+1}\right)\\ &> \frac{\ell+1}{2N-\ell+1}\cdot\frac{(2N+\ell-1)M(M+1)}{2d(\ell+1)} = \frac{\sigma_M}{d}.\end{aligned} \tag{2}$$

Then, we can choose a rational number ϵ (> 0) such that

$$\frac{d(\sum_{j=1}^{q}\omega_j - M - 1 - \frac{\lambda\rho}{d}) - \sigma_M}{d(\sigma_{M+1}+\frac{\rho}{d}) + \tau_M} > \epsilon > \frac{d(\sum_{j=1}^{q}\omega_j - M - 1 - \frac{\lambda\rho}{d}) - \sigma_M}{\frac{1}{mq} + d(\sigma_{M+1}+\frac{\rho}{d}) + \tau_M}.$$

We define the following numbers

$$\begin{aligned}\beta &:= d\left(\sum_{j=1}^{q}\omega_j - M - 1 - \frac{\lambda\rho}{d} - \epsilon\left(\sigma_{M+1}+\frac{\rho}{d}\right)\right) > \sigma_M + \epsilon\tau_M,\\ \rho &:= \frac{1}{\beta}(\sigma_M + \epsilon\tau_M),\\ \rho^* &:= \frac{1}{(1-\rho)\beta} = \frac{1}{d(\sum_{j=1}^{q}\omega_j - M - 1 - \frac{\lambda\rho}{d}) - \sigma_M - \epsilon(d\sigma_{M+1}+\rho+\tau_M)}.\end{aligned}$$

It is clear that $0 < \rho < 1$ and $\frac{\epsilon\rho^*}{mq} > 1$.

We consider a set

$$S' = \{a \in S; \psi(F^i)_{jp}(a) \neq 0, h(a) \neq 0\ \forall 1 \leq i \leq m; j = 1,\ldots,q; p = 0,\ldots,M\}$$

and define a new pseudo-metric on S' as follows

$$d\tau^2 = |\xi|^{\frac{2(1+\beta\rho\rho^*)}{m}}\left(\frac{1}{|h|^{\lambda+\epsilon}}\prod_{i=1}^{m}\frac{\prod_{j=1}^{q}|Q_j(F^i)|^{\omega_j}}{|F^i_M|^{1+\epsilon}\prod_{j,p}|\psi(F^i)_{jp}|^{\frac{\epsilon}{q}}}\right)^{\frac{2\rho^*}{m}}|dz|^2.$$

Since $Q_j(F^i), F^i_M, \psi(F^i)_{jp}$ $(1\le j\le q)$ and h are all holomorphic functions on S', $d\tau^2$ is flat on S'. We now show that $d\tau^2$ is complete on S'.

Indeed, suppose contrarily that S' is not complete with $d\tau^2$, there is a divergent curve $\gamma : [0,1)\to S'$ with finite length. Then, as $t\to 1$ there are only two cases: either $\gamma(t)$ tends to a point a with

$$(h\prod_{j=1}^{q}\prod_{p=0}^{M}\psi(F^i)_{jp})(a)=0$$

or else $\gamma(t)$ tends to the boundary of S.

For the first case, by Theorem 2.4, we have

$$\begin{aligned}\nu_{d\tau}(a)\le -\Big(&\sum_{i=1}^{m}\nu_{F^i_M}(a)-\sum_{i=1}^{m}\sum_{j=1}^{q}\omega_j\nu_{Q_j(F^i)}(a)+\lambda\nu_h(a)\\&+\big(\epsilon\sum_{i=1}^{m}\nu_{F^i_M}(a)+\epsilon\nu_h(a)+\frac{\epsilon}{q}\sum_{i=1}^{m}\sum_{j,p}\nu_{\psi(F^i)_{jp}}(a)\big)\Big)\frac{\rho^*}{m}\\&\le -\frac{\epsilon\rho^*}{m}\left(\sum_{i=1}^{m}\nu_{F^i_M}(a)+\nu_h(a)\right)-\frac{\epsilon\rho^*}{mq}\sum_{i=1}^{m}\sum_{j,p}\nu_{\psi(F^i)_{jp}}(a)\le -\frac{\epsilon\rho^*}{mq}.\end{aligned}$$

Then, there is a positive constant C such that

$$|d\tau|\ge\frac{C}{|z-a|^{\frac{\epsilon\rho^*}{mq}}}|dz|$$

in a neighborhood of a. Then we get

$$L_{d\tau}(\gamma)=\int_0^1\|\gamma'(t)\|_\tau dt=\int_\gamma d\tau\ge\int_\gamma\frac{C}{|z-a|^{\frac{\epsilon\rho^*}{mq}}}|dz|=+\infty$$

($\gamma(t)$ tends to a as $t\to 1$). This is a contradiction. Then, the second case must occur, i.e., $\gamma(t)$ tends to the boundary of S as $t\to 1$.

Take a disk Δ (in the metric induced by $d\tau^2$) around $\gamma(0)$. Since $d\tau$ is flat, by Lemma 3.3, Δ is isometric to an ordinary disk in the plane. Let $\Phi:\Delta(r)=\{|\omega|<r\}\to\Delta$ be this isometric with $\Phi(0)=\gamma(0)$. Extend Φ as a local isometric into S' to a the largest disk possible $\Delta(R)=\{|\omega|<R\}$, and denoted again by Φ this extension (for simplicity, we may consider Φ as the exponential map). Since Φ cannot be extended to a larger disk, it must be hold that the image of Φ goes to the boundary of S'. But, this image cannot go to points z_0 of S with $h(z_0)\prod_{i=1}^m\Big(F^i_M(z_0)\prod_{j,p}\psi(F^i)_{jp}|(z_0)\Big)=0$, since we have already shown that $\gamma(0)$ is infinitely far away in the metric $d\tau^2$ with respect to these points. Then the image of Φ must go to the boundary S. Hence, by again Lemma 3.3, there exists a point w_0 with $|w_0|=R$ so that $\Gamma=\Phi(\overline{0,w_0})$ is a divergent curve on S.

Our goal now is to show that Γ has finite length in the original metric ds^2 on S, contradicting the completeness of S. Let $g^i:=f^i\circ\Phi:\Delta(R)\to V\subset\mathbb{P}^n(\mathbb{C})$ be a holomorphic map which is nondegenerate over $I_d(V)$. Then g^i have a reduced representation

$$G^i=(g^i_0,\dots,g^i_n),$$

where $g^i_j = f^i_j \circ \Phi$ $(1 \leq i \leq m, 0 \leq j \leq n)$. Hence, we have:

$$\begin{aligned}
\Phi^* ds^2 &= 2|\xi \circ \Phi|^{2/m} \prod_{i=1}^{m} \|F^i \circ \Phi\|^{2/m} |\Phi^* dz|^2 \\
&= 2|\xi \circ \Phi|^{2/m} \left(\prod_{i=1}^{m} \|G^i\|^{2/m} \right) \left| \frac{d(z \circ \Phi)}{dw} \right|^2 |dw|^2, \\
G^i_M &= (F^i \circ \Phi)_M = F^i_M \circ \Phi \cdot \left(\frac{d(z \circ \Phi)}{dw} \right)^{\sigma_M}, \\
\psi(G^i)_{jp} &= \psi(F^i \circ \Phi)_{jp} = \psi(F^i)_{jp} \cdot \left(\frac{d(z \circ \Phi)}{dw} \right)^{\sigma_p}, (0 \leq p \leq M).
\end{aligned}$$

On the other hand, since Φ is locally isometric,

$$\begin{aligned}
&|dw| = |\Phi^* d\tau| \\
&= |\xi \circ \Phi|^{\frac{1+\beta\rho\rho^*}{m}} \left(\frac{1}{|h \circ \Phi|^{\lambda+\epsilon}} \prod_{i=1}^{m} \frac{\prod_j |Q_j(F^i) \circ \Phi|^{\omega_j}}{|F^i_M \circ \Phi|^{1+\epsilon} \prod_{j,p} |\psi(F^i)_{jp} \circ \Phi|^{\epsilon/q}} \right)^{\rho^*/m} \left| \frac{d(z \circ \Phi)}{dw} \right| \cdot |dw| \\
&= |\xi \circ \Phi|^{\frac{1+\beta\rho\rho^*}{m}} \left(\frac{1}{|h \circ \Phi|^{\lambda+\epsilon}} \prod_{i=1}^{m} \frac{\prod_j |Q_j(G^i)|^{\omega_j}}{|G^i_M|^{1+\epsilon} \prod_{j,p} |\psi(G^i)_{jp}|^{\epsilon/q}} \right)^{\rho^*/m} \left| \frac{d(z \circ \Phi)}{dw} \right|^{1+\beta\rho\rho^*} \cdot |dw|
\end{aligned}$$

(because $1 + \rho^*(\sigma_M + \epsilon\tau_M) = 1 + \beta\rho\rho^*$). This implies that

$$\begin{aligned}
\left| \frac{d(z \circ \Phi)}{dw} \right| &= |\xi \circ \Phi|^{-\frac{1}{m}} \left(|h \circ \Phi|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|G^i_M|^{1+\epsilon} \prod_{j,p} |\psi(G^i)_{jp}|^{\epsilon/q}}{\prod_j |Q_j(G^i)|^{\omega_j}} \right)^{\frac{\rho^*}{m(1+\beta\rho\rho^*)}} \\
&\leq |\xi \circ \Phi|^{-\frac{1}{m}} \left(|h \circ \Phi|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|G^i_M|^{1+\epsilon} \prod_{j,p} |G^i_p(Q_j)|^{\epsilon/q}}{\prod_j |Q_j(G^i)|^{\omega_j}} \right)^{\frac{\rho^*}{m(1+\beta\rho\rho^*)}} \\
&= |\xi \circ \Phi|^{-\frac{1}{m}} \left(|h \circ \Phi|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|G^i_M|^{1+\epsilon} \prod_{j,p} |G^i_p(Q_j)|^{\epsilon/q}}{\prod_j^q |Q_j(G^i)|^{\omega_j}} \right)^{\frac{1}{m\beta}}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\Phi^* ds &\leq \sqrt{2} \prod_{i=1}^{m} \|G^i\|^{\frac{1}{m}} \left(|h \circ \Phi|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|G^i_M|^{1+\epsilon} \prod_{j,p} |G^i_p(Q_j)|^{\epsilon/q}}{\prod_{j=1}^q |Q_j(G^i)|^{\omega_j}} \right)^{\frac{1}{m\beta}} |dw| \\
&= \sqrt{2} \left(|h \circ \Phi|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{\|G^i\|^l |G^i_M|^{1+\epsilon} \prod_{j,p} |G^i_p(Q_j)|^{\epsilon/q}}{\prod_{j=1}^q |Q_j(G^i)|^{\omega_j}} \right)^{\frac{1}{m\beta}} |dw| \\
&\leq C_1 \left(|h \circ \Phi|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|G^i_0|^{\frac{l}{d}} |G^i_M|^{1+\epsilon} \prod_{j,p} |G^i_p(Q_j)|^{\epsilon/q}}{\prod_{j=1}^q |Q_j(G^i)|^{\omega_j}} \right)^{\frac{1}{m\beta}} |dw|.
\end{aligned}$$

with a positive constant C_1. We note that $\frac{\beta}{d} = \sum_{j=1}^{q} \omega_j - M - 1 - \frac{\lambda\rho}{d} - \epsilon\left(\sigma_{M+1} + \frac{\rho}{d}\right)$. Then the inequality (2) yields that the conditions of Lemma 2.5 are satisfied. Then, by applying Lemma 2.5, we have

$$\Phi^* ds \leq C_2 \left(\frac{2R}{R^2 - |w|^2}\right)^{\rho} |dw|$$

for some positive constant C_2. Also, we have that $0 < \rho < 1$. Then

$$L_{ds^2}(\Gamma) \leq \int_{\Gamma} ds = \int_{\overline{0, w_0}} \Phi^* ds \leq C_2 \cdot \int_0^R \left(\frac{2R}{R^2 - |w|^2}\right)^{\rho} |dw| < +\infty.$$

This contradicts the assumption of completeness of S with respect to ds^2. Thus, $d\tau^2$ is complete on S'.

Then, we note that the metric $d\tau^2$ on S' is flat. Then by a theorem of Huber (cf. [2, Theorem 13, p.61]), the fact that S' has finite total curvature (with respect to $d\tau^2$) implies that S' is finitely connected. This means that there is a compact subregion of S' whose complement is the union of a finite number of doubly-connected regions. Therefore, the functions $|h| \prod_{p=0}^{M} \prod_{j=1}^{q} |\psi(G_z)_{jp}|$ must have only a finite number of zeros, and the original surface S is finitely connected. Due to Osserman (cf. [7, Theorem 2.1]), each annular ends of S', and hence of S, is conformally equivalent to a punctured disk. Thus, the Riemann surface S must be conformally equivalent to a compact surface $\overline{S}$ punctured at a finite number of points $P_1, \ldots, P_r$. Then, there are disjoint neighborhoods U_i of P_i $(1 \leq i \leq r)$ in $\overline{S}$ and biholomorphic maps $\phi_i : U_i \to \Delta$ with $\phi_i(P_i) = 0$. Note that, the Poincare-metric on $\Delta^* = \Delta \setminus \{0\}$ is given by $d\sigma^2_{\Delta^*} = \dfrac{4|dw|^2}{|w|^2 \log^2 |w|^2}$, where w is the complex coordinate on Δ.

As we known that the universal covering surface of S is biholomorphic to $\mathbb{C}$ or a disk in $\mathbb{C}$. If the universal covering of S is biholomorphic to $\mathbb{C}$ (denote by $\tilde{\Phi} : \mathbb{C} \to S$ this universal covering mapping), then from the assumption that

$$\lambda\nu_h + \sum_{i=1}^{m} \nu_{F^i_M} \geq \sum_{i=1}^{m} \sum_{j=1}^{q} \omega_j \nu_{Q_j(F^i)} \text{ and } |h| \leq \prod_{i=1}^{m} \|F^i\|^{\rho},$$

we have

$$\lambda\rho \sum_{i=1}^{m} T_{f^i \circ \tilde{\Phi}} \sum_{j=1}^{q} N(r, \nu_{h \circ \tilde{\Phi}}) \geq \sum_{i=1}^{m} \left(\sum_{j=1}^{q} \omega_j N(r, \nu_{Q_j(F^i \circ \tilde{\Phi})}) - N(r, \nu_{F^i_M \circ \Phi}) \right),$$

where $T_f(r)$ is the characteristic function of the mapping $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ and $N(r, \nu)$ is the counting function of the divisor ν on $\mathbb{C}$ (see [9] for the definitions). Using the second main theorem (Theorem 1.1 in [9]), we have

$$\| \ \lambda\rho \sum_{i=1}^{m} T_{f^i \circ \tilde{\Phi}} \geq \sum_{i=1}^{m} (\sum_{j=1}^{q} N^{[M]}(r, \nu_{Q_j(F^i \circ \tilde{\Phi})}) \geq \left(q - \frac{(2N - \ell + 1)(M + 1)}{\ell + 1}\right) \sum_{i=1}^{m} T_{f^i \circ \tilde{\Phi}}.$$

Here, the symbol "$\|$" means the inequalities hold for all $r \in [1, +\infty)$ outside a finite Borel measure set E. Letting $r \to +\infty$ ($r \notin E$), we get

$$\lambda\rho \geq q - \frac{(2N - \ell + 1)(M+1)}{\ell + 1}$$

and arrive at a contradiction.

Then, we only consider the case where the universal covering surface of S is biholomorphic to the unit disk Δ in $\mathbb{C}$. Let $\tilde{\Phi} : \Delta \to S$ be this holomorphic universal covering. Consider the following metric

$$d\tilde{\tau}^2 = \eta^2 |dz|^2,$$

where

$$\eta = \left(|h|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|F_0^i|^{\gamma - \epsilon(\sigma_{M+1} + \frac{\rho}{d})} |F_M^i|^{1+\epsilon} \prod_{j,p} |F_p^i(Q_j)|^{\epsilon/q}}{\prod_{j=1}^{q} |Q_j(F^i)|^{\omega_j}} \right)^{\frac{1}{m(\sigma_M + \epsilon\tau_M)}}.$$

It is obvious that $d\tilde{\tau}^2$ is continuous on $S \setminus \bigcup_{j=1}^{q} (f^i)^{-1}(Q_j)$. Take a point a such that $\prod_{j=1}^{q} Q_j(F^i)(a) = 0$. From the assumption, we have

$$d\tilde{\tau}(a) \geq \frac{1}{m(\sigma_M + \epsilon\tau_M)} \left(\lambda\nu_h(a) + \sum_{i=1}^{m} \nu_{F_M^i}(a) - \sum_{i=1}^{m} \sum_{j=1}^{q} \omega_j \nu_{Q_j(F^i)}(a) \right) \geq 0.$$

Therefore $d\tilde{\tau}$ is continuous at a. This yields that $d\tilde{\tau}$ is a continuous pseudo-metric on S. Now, from Lemma 3.1, we see that $d\tau^2$ has strictly negative curvature on S Hence, by the decreasing distance property, we have

$$\Phi^* d\tau^2 \leq d\sigma_\Delta^2 \leq C_3 \cdot (\Phi \circ \phi_i^{-1})^* d\sigma_{\Delta^*}^2 \ (1 \leq i \leq r)$$

for some positive constant C_3. This implies that

$$\int_{U_i} \Omega_{d\tau^2} \leq \int_{\Phi^{-1}(U_i)} \Phi^* \Omega_{\sigma_\Delta^2} \leq l_0 C_3 \int_{\Delta^*} \Omega_{d\sigma_{\Delta^*}^2} < \infty.$$

where l_0 is the number of the sheets of the covering Φ. Then, it yields that

$$\int_S \Omega_{d\tau^2} \leq \int_{S \setminus \bigcup_{i=1}^{r} U_i} \Omega_{d\tau^2} + l_0 C_3 r \int_{\Delta^*} \Omega_{d\sigma_{\Delta^*}^2} < \infty.$$

Now, denote by ds^2 the original metric on S. Similar as (1), we have

$$\mathrm{dd^c} \log \eta \geq \frac{\gamma - \epsilon(\sigma_{M+1} + \frac{\rho}{d})}{\sigma_M + \epsilon\tau_M} \frac{d}{m} \sum_{i=1}^{m} \Omega_{f_i}.$$

Then there is a subharmonic function v such that

$$\begin{aligned} \lambda^2 |dz|^2 &= e^v |\xi|^{\frac{2}{m}} \left(\prod_{i=1}^{m} \|F^i\|^{2 \frac{\gamma - \epsilon(\sigma_{M+1} + \frac{\rho}{d})}{m(\sigma_M + \epsilon\tau_M)}} \right) |dz|^2 \\ &= e^v \left(\prod_{i=1}^{m} \|F^i\|^{2 \frac{\gamma - \sigma_M - \epsilon\tau_{M+1}}{m(\sigma_M + \epsilon\tau_M)}} \right) |\xi|^{\frac{2}{m}} \left(\prod_{i=1}^{m} \|F^i\|^{\frac{2}{m}} \right) |dz|^2 \\ &= e^w ds^2 \end{aligned}$$

for a subharmonic function w on S. Since S is complete with respect to ds^2, applying a result of S. T. Yau [14] and L. Karp [6] we have

$$\int_S \Omega_{d\tau^2} = \int_S e^w \Omega_{ds^2} = +\infty.$$

This contradiction completes the proof of the theorem. $\square$

4 Uniqueness theorems for Gauss maps

In this section, we will prove main theorems of this paper. Firstly, we prove the following.

Lemma 4.1. *Let S be an open Riemann surface and V be a ℓ-dimension projective subvariety of $\mathbb{P}^n(\mathbb{C})$. Let $Q_1, \dots, Q_q$ be q hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in $N-$subgeneral position with respect to V and $d = \operatorname{lcm}(\deg Q_1, \dots, \deg Q_q)$. Let f^1, f^2 be holomorphic maps from S into V such that*

(1) $\bigcap_{j=0}^k (f^1)^{-1}(Q_{i_j}) = \emptyset$ for every $1 \le j_0 < \cdots < j_k \le q$,

(2) $f^1 = f^2$ on $\bigcup_{j=1}^q \big((f^1)^{-1}(Q_j) \cup (f^2)^{-1}(Q_j)\big)$.

If f^1 is nondegenerate over $I_d(V)$, S is complete with a metric $ds^2 = |\xi|^2\|F^1\|^2|dz^2|$, where ξ is a non-vanishing holomorphic function, z is a conformal coordinate on S, F^1 is a reduced presentation of f^1, and

$$q > \frac{2N-\ell+1}{\ell+1}\left(M+1+\sigma_M - \sigma_{M-\min\{k,\ell\}} + \frac{M(M+1)}{2d}\right)$$

then f^2 is nondegenerate over $I_d(V)$.

Proof. Let $F^i = (f_0^i, \dots, f_n^i)$ be reduce representations of f^i $(i = 1, 2)$. Suppose contrarily that f^2 is degenerate over $I_d(V)$. Then there exists a hypersurface Q of degree d such that $V \not\subset Q$, $Q(F^2) \equiv 0$. By the assumption that f^1 is nondegenerate over $I_d(V)$, we have $Q(F^1) \not\equiv 0$. Since $f^1 = f^2$ on $\bigcup_{i=1}^q Q_i$, we have $Q(F^1) = 0$ on $\bigcup_{i=1}^q (f^1)^{-1}Q_i$. Therefore, setting $k' = \min\{k, \ell\}$ and $h = Q(F^1)$, from Theorem 2.4 we have

$$(\sigma_M - \sigma_{M-k'})\nu_h(a) + \nu_{F^1_M}(a) \ge \sum_{j=1}^{k'} \nu_{Q_j(F^1)}(a) \ge \sum_{j=1}^{q} \omega_j \nu_{Q_j(F^1)}(a).$$

Also, it is clear that $|h| \le \|F^1\|^d$. Applying Theorem 3.4, we have

$$q \ge \frac{2N-\ell+1}{\ell+1}\left(M+1+\sigma_M - \sigma_{M-k'} + \frac{M(M+1)}{2d}\right).$$

This contradiction completes the proof of the lemma. $\square$

Proof of Theorem 1.1. Without loss of generality, we may assume that $\deg Q_j = d$ for all $1 \le j \le q$. We may suppose that $f^1(S) \not\subset Q_j$ for all $j \in \{1, \dots, q\}$ (otherwise $f^1 = f^2$). Let z be a conformal coordinate on S^1 and $F^i = (f_0^i, \dots, f_n^i)$ be the reduce representation of f^i for each $i \in \{1, 2\}$. Since Φ is a conformal diffeomorphism, there exists a non-vanishing holomorphic function ξ such that $ds^2 = \|F^1\|^2|dz^2| = |\xi|^2\|F^2\|^2|dz^2|$. We have ds^2 is complete on S^1. Also, from the proof of Lemma 4.1, we see that $f^2(S^1) \subset V$ and f^2 is nondegenerate over $I_d(V)$.

Suppose contrarily that $f^1 \not\equiv f^2$. Then there exists $0 \le i < j \le n$ such that

$$h := f_i^1 f_j^2 - f_1^j f_i^2 \not\equiv 0.$$

It is clear that $\nu_h \ge \nu^{[1]}_{\prod_{j=1}^q Q_j(F^i)}$ for each $i \in \{1, 2\}$.

From Theorem 2.4, we have

$$2(\sigma_M - \sigma_{M-\min\{k,\ell\}})\nu_h + \sum_{i=1}^{2} \nu_{F^i_M} \ge \sum_{i=1}^{2}\sum_{j=1}^{q} \nu_{Q_j(F^i)}.$$

Note that $|h| \le \|F^1\| \cdot \|F^2\|$. By applying Theorem 3.4, we have

$$q \le \frac{2N - \ell + 1}{\ell + 1}\left(M + 1 + \frac{2(\sigma_M - \sigma_{M-\min\{k,\ell\}})}{d} + \frac{M(M+1)}{2d}\right).$$

This contradiction completes the proof of the theorem. □

Proof of Theorem 1.2. Let z be a conformal coordinate on S^1 and F^i be the reduce representation of f^i for each $i \in \{1, 2\}$. Since Φ is a conformal diffeomorphism, there exists a non-vanishing holomorphic function ξ such that $ds^2 = \|F^1\|^2|dz^2| = |\xi|^2\|F^2\|^2|dz^2| = |\xi| \cdot \|F^1\| \cdot \|F^2\||dz^2|$. Note that, ds^2 is complete on S^1.

Suppose contrarily that the theorem does not hold. Consider the simple graph $\mathcal{G}$ with the set of vertices $\{1, \dots, q\}$ and the set of edges consisting of all pairs $\{i, j\}$ such that $Q_i(F^1)Q_j(F^2) - Q_j(F^1)Q_i(F^2) \not\equiv 0$. The supposition implies that the order of each vertex does not exceed $[q/2]$. Then, by Dirac's theorem, there is a Hamilton cycle $i_1, \dots, i_q, i_{q+1}$, where $i_{q+1} = i_1$. We set $u_j := i_{j+1}$ if $j < q$ and $u_q := i_1$. Then we have

$$h := \prod_{j=1}^{q} (Q_{i_j}(F^1)Q_{u_j}(F^2) - Q_{u_j}(F^1)Q_{i_j}(F^2)) \not\equiv 0.$$

It is clear that $|h| \le \|F^1\|^{dq}\|F^2\|^{dq}$.

For each point $a \in \bigcup_{j=1}^q (f^1)^{-1}(Q_j)$, take a subset $I_1 \subset \{1, \dots, q\}$ such that $\sharp I = N + 1$ and $Q_j(F^1)(a) \ne 0$ for every $j \notin I_1$. Then there is a subset $I_2 \subset I_1$ such that $\sharp I_2 = \operatorname{rank}_{\mathbb{C}}\{[Q_j]; j \in I_2\} = \ell + 1$ and

$$\sum_{i \in I_2} (\nu_{Q_i(F^1)}(a) + \nu_{Q_i(F^2)}(a)) \ge \sum_{i \in I_1} \omega_i(\nu_{Q_i(F^1)}(a) + \nu_{Q_i(F^2)}(a)).$$

Then, there exists a subset $I \subset I_2$ such that $\sharp I = t$ and $Q_j(F^1)(a) \neq 0$ for every $j \in I_2 \setminus I$. Hence, we have

$$\sum_{i\in I}(\nu_{Q_i(F^1)}(a) + \nu_{Q_i(F^2)}(a)) \geq \sum_{i=1}^{q}\omega_i(\nu_{Q_i(F^1)}(a) + \nu_{Q_i(F^2)}(a)).$$

Then, we have

$$\begin{aligned}\nu_h(a) &\geq 2\sum_{i=1}^{t}\min\{\nu_{Q_i(F^1)}(a), \nu_{Q_i(F^2)}(a)\} + (q-2t)\\ &\geq 2\sum_{i=1}^{t}\big(\min\{\nu_{Q_i(F^1)}(a), M\} + \min\{\nu_{Q_i(F^2)}(a), M\} - M\big) + (q-2t)\\ &= 2\sum_{i=1}^{t}\big(\min\{\nu_{Q_i(F^1)}(a), M\} + \min\{\nu_{Q_i(F^2)}(a), M\}\big) + (q-2(M+1)t)\\ &\geq \frac{q+2(M-1)t}{2Mt}\sum_{i=1}^{t}\big(\min\{\nu_{Q_i(F^1)}(a), M\} + \min\{\nu_{Q_i(F^2)}(a), M\}\big).\end{aligned}$$

Also, by usual arguments we have

$$\nu_{F^1_M F^2_M}(a) \geq \sum_{j=1}^{2}\sum_{i=1}^{t}\max\{0, \nu_{Q_i(F^j)}(a) - M\}.$$

Therefore, we have the following estimate:

$$\begin{aligned}\frac{2Mt}{q+2(M-1)t}\nu_h(a) + \nu_{F^1_M F^2_M}(a) &\geq \sum_{i=1}^{t}(\nu_{Q_i(F^1)}(a) + \nu_{Q_i(F^2)}(a))\\ &\geq \sum_{i=1}^{q}\omega_i(\nu_{Q_i(F^1)}(a) + \nu_{Q_i(F^2)}(a))\end{aligned}$$

By Theorem 3.4, we have

$$\begin{aligned}q &\leq \frac{2N-\ell+1}{\ell+1}\left(M+1+\frac{2Mtq}{(q+2(M-1)t)d} + \frac{M(M+1)}{2d}\right)\\ &= \frac{2N-\ell+1}{\ell+1}\left(M+1+\frac{2Mkq}{(q+2(M-1)k)d} + \frac{M(M+1)}{2d}\right)\end{aligned}$$

and arrive at a contradiction. This completes the proof of the theorem. □

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