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The zero-divisor graph of the edge-ring of a simple graph

by

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Abstract

We study the zero-divisor graph for a commutative ring R denoted by $\Gamma(R)$, this for the ring of edges $R(\mathcal{G})$ of a simple graph $\mathcal{G}$. Based on the graph $\Gamma(R(\mathcal{G}))$, we characterize the ring of edges with the graph invariants diameter and girth. Moreover, for this family of graphs, we compute the clique number and the chromatic number, obtaining that this family of graphs is weakly perfect.

Key Words: Monomial ideal, edge ideal, Stanley decompositions, Stanley depth.
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1 Introduction

In algebraic combinatorics there is a great interest in the association of a graphical object with an algebraic structure, providing a broad study between combinatorial and algebraic invariants. Moreover, the study of graphs associated with rings has become one of the active areas in this field. In the literature we can find examples where combinatorial methods are used to obtain algebraic results in ring theory, we can mention ([1], [2],[3],[4] and [5]).

Let us begin by recalling the two concepts discussed in this article, the ring of edges for a simple graph $\mathcal{G}$ and the graph of zero divisors for a commutative ring R, each of which will be discussed below.

Given a simple graph $\mathcal{G}$ consisting of a finite set of vertices $V = \{x_1, x_2, \dots, x_n\}$ and a set of edges. The set of vertices and the set of edges of $\mathcal{G}$ are denoted by $V(\mathcal{G})$ and $E(\mathcal{G})$, respectively.

Considering the vertices of the graph $\mathcal{G}$, we define $A = K[x_1, x_2, \dots, x_n]$ the ring of polynomials over a field K and the ideal of the edges of $\mathcal{G}$, denoted by $I(\mathcal{G})$ that is the ideal of A generated for all the monomials $x_e = \prod_{x_i \in e} x_i$ such that $e \in E(\mathcal{G})$. The map $\mathcal{G} \longrightarrow I(\mathcal{G})$ is a one-to-one correspondence between the family of simple graphs and the family of square-free monomial ideals. The quotient ring $R = A/I(\mathcal{G})$ is called the ring of edges. The ideals of edges have been introduced and studied in [8, 9].

If R is a commutative ring, we define the simple graph $\Gamma(R) = (V, E)$ where $V = Z(R) \neq \{0\}$, where $Z(R)$ symbolizes the set of zero divisors of the commutative ring R. Now for $x, y \in V$ we have that $\{x, y\} \in E$ if and only if $xy = 0$. The definition that will be used throughout this article is the one presented by Anderson and Livinstone in [2]. After they modified the original definition proposed by I. Beck in 1988 [4]. This graph has been studied extensively in recent years.

In this paper we analyze the properties of the zero-divisor graph of the ring of edges for a simple graph $\mathcal{G}$, which would be denoted by $\Gamma(R(\mathcal{G}))$.

We need to recall some concepts from graph theory. By an undirected graph G we mean an ordered pair $G = (V, E)$, where V is a nonempty set of vertices and E is a set of edges. We say that G is connected if there exists a path between any two vertices. The subgraph induced by $X \subseteq V$ in G, denoted $G[X]$, is the graph which satisfies that if two vertices in X are adjacent in $G[X]$, if and only if are they adjacent in G. In a graph G, we define a path p as a sequence of edges $x_1 \sim x_2, \ldots, x_{k-1} \sim x_k$ of the graph G. We say that G is connected if there is a path between any two vertices. For vertices x and y in G, the distance between x and y, denoted by $d(x, y)$, is the length of the shortest path connecting x to y. From this definition it follows that $d(x, x) = 0$. If no such path exists, we say that $d(x, y) = \infty$. The diameter of G, for which we introduce the following notation $diam(G)$, is defined by $diam(G) = sup\{d(x, y) : x, y \in V\}$. A cycle of length n is a path $p = x_1 \sim x_2 \sim \cdots \sim x_n \sim x_1$, where $x_i \neq x_j$ if $i \neq j$. We define the girth of G, denoted by $girth(G)$, as the length of the shortest cycle in G if G contains a cycle, otherwise $girth(G) = \infty$. The order of G is defined as $|G| := |E(G)|$. It is clear that $girth(G^{'}) \geq girth(G)$ if $G^{'}$ is an induced subgraph of G, but it is not true that there is a relation between $diam(G^{'})$ and $diam(G)$.

A graph G is complete if any two vertices are adjacent, the complete graph with n vertices would be denoted by K_n. A graph is bipartite if there exists a pair of disjoint sets A, B such that $V(G) = A \cup B$, where any edge of G connects vertices of A with vertices of B. If all vertices of A are adjacent to all vertices of B, then the graph G is a complete bipartite graph. If one of the sets A or B in a complete bipartite graph is singular, then we say that the graph G is a star graph. We denote the complete bipartite graph by $K_{m,n}$, where $|A| = m$ and $|B| = n$. Thus, a star graph is a $K_{1,n}$ complete bipartite graph.

More generally, the graph G is complete r-partite if G is the disjoint union of r nonempty vertex sets, and two distinct vertices are adjacent if and only if they are in distinct vertex sets. A complete subgraph of G is called a clique. The clique number is denoted by $\omega(G)$, which is the largest integer $r \geq 1$ such that $K_r \subseteq G$ (if $K_r \subseteq G$ for all integers $r \geq 1$, then we define $\omega(G) = \infty$). On the other hand, the chromatic number of G, denoted by $\chi(G)$, is the minimum number of colors needed to color the vertices of G such that no two adjacent vertices have the same color. Obviously, $\omega(G) \leq \chi(G)$. Two graphs G_1, G_2 are isomorphic if there exists a bijection f between the vertices of G_1 and the vertices of G_2 such that x and y are adjacent in G_1 if and only if $f(x)$ and $f(y)$ are adjacent in G_2.

Formally, a vertex cover C of a graph $G = (V, E)$ is a subset of V such that $\{u, v\} \in E$ then $u \in C$ or $v \in C$, i.e., it is a set of vertices C where each edge has at least one terminal vertex in the vertex cover C. Such a set is said to cover the edges of G. In Section 2 we deal with the characterization of the graph $\Gamma(R(\mathcal{G}))$ in terms of the girth. In Section 3, we classify the graph $\Gamma(R(\mathcal{G}))$ by its diameter. For section 4, we study the clique number of $\Gamma(R(\mathcal{G}))$ and find the relation that this invariant has with respect to the graph $\mathcal{G}$. Finally, in section 5; we focus our efforts on computing the chromatic number of $\Gamma(R(\mathcal{G}))$, which coincides with the cover number of $\mathcal{G}$. It is concluded that this family of graphs satisfies the property of being weakly perfect.

2 Girth for the graph $\Gamma(R(\mathcal{G}))$

Proposition 2.1. *Let $\mathcal{G}$ be a simple graph, then girth$(\Gamma(R(\mathcal{G}))) \leq 4$.*

Proof. Let $\varphi : K[x_1, \ldots, x_n] \longrightarrow R(\mathcal{G})$ be the canonical homomorphism.

$$p(x) \mapsto \overline{p(x)} \ mod\ I(\mathcal{G}).$$

Then, since there exists $\{a, b\} \in E$, it follows that $a \sim b \sim a^2 \sim b^2 \sim a$ is a 4-cycle in $\Gamma(R(\mathcal{G}))$ □

Proposition 2.2. *Given an arbitrary simple graph $\mathcal{G}$, it follows that girth$(\Gamma(R(\mathcal{G}))) \leqslant$ girth$(\mathcal{G})$.*

Proof. We can assume that $girth(\mathcal{G}) = m$ is finite. Consider the m-cycle $x_1 \sim x_2 \sim \cdots \sim x_m \sim x_1$. Then we have that $x_1 \sim x_2 \sim \cdots \sim x_m \sim x_1$ is an m-cycle in $\Gamma(R(\mathcal{G}))$, since every vertex of $\mathcal{G}$ belongs to $Z(R(\mathcal{G}))$ and $\{x_i, x_j\} \in E$ implies $x_i x_j = 0 \in R(\mathcal{G})$. □

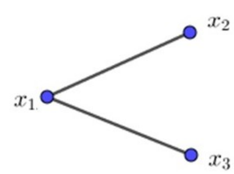
Corollary 2.3. *Let $\mathcal{G}$ be a simple graph such that it has a 3-cycle, then girth$(\Gamma(R(\mathcal{G}))) = 3$.*

Proof. Let $\mathcal{G}$ be a simple graph such that $\mathcal{G}$ has a 3-cycle, by Proposition 2.2 we have that $girth(\Gamma(R(\mathcal{G}))) = 3$. □

Example 2.4. *First consider $\mathcal{G}$, the simple graph formed by the pair $(V = \{x_1, x_2\}, E = \{\{x_1, x_2\}\})$, i.e,*

So $I(\mathcal{G}) = \langle x_1 \rangle \cap \langle x_2 \rangle$ with $Z(R(\mathcal{G})) = \langle x_1 \rangle \cup \langle x_2 \rangle$. Suppose there exist $a, b, c \in Z(R(\mathcal{G}))^*$ such that $a \sim b \sim c \sim a$ form a 3-cycle, without loss of generality we can assume that $a \in \langle x_1 \rangle$, which implies that $b \in \langle x_2 \rangle$, analogously $c \in \langle x_1 \rangle$ and $a \in \langle x_2 \rangle$, which is a contradiction since $a \in \langle x_1 \rangle \cap \langle x_2 \rangle$ implies that $a = 0$. Therefore girth$(\Gamma(R(\mathcal{G}))) = 4$.

Example 2.5. *Now let $\mathcal{G}$ be the simple graph given by the pair $(V = \{x_1, x_2, x_3\}, E = \{\{x_1, x_2\}, \{x_1, x_3\}\})$, i.e,*



Let us note that $I(\mathcal{G}) = \langle x_1 \rangle \cap \langle x_2, x_3 \rangle$ with a similar argument as in Example 2.4, we have that girth$(\Gamma(R(\mathcal{G}))) = 4$.

What was shown in Example 2.4 and Example 2.5 motivates the following proposition.

Proposition 2.6. *Let $\mathcal{G}$ be a simple graph with exactly two covers by minimal vertices. Then* $girth(\Gamma(R(\mathcal{G}))) = 4$.

Proof. Let C_1, C_2 be the minimal vertex covers of $\mathcal{G}$. Then $I(\mathcal{G}) = \langle C_1 \rangle \cap \langle C_2 \rangle$. Suppose there exist $\overline{a}, \overline{b}, \overline{c} \in Z(R(\mathcal{G}))^*$ such that $\overline{a} \sim \overline{b} \sim \overline{c} \sim \overline{a}$ is a 3-cycle. Without loss of generality, we consider that $\overline{a} \in \langle C_1 \rangle$, which necessarily implies that $\overline{b} \in \langle C_2 \rangle$, since $Z(R(\mathcal{G})) = \langle C_1 \rangle \cup \langle C_2 \rangle$. Proceeding in this way, we get that $\overline{c} \in \langle C_1 \rangle$ and $\overline{a} \in \langle C_2 \rangle$. But $a \in \langle C_1 \rangle \cap \langle C_2 \rangle$ implies that $\overline{a} = \overline{0}$, which contradicts that $a \in Z(R(\mathcal{G}))^*$. Hence $girth(\Gamma(R(\mathcal{G}))) = 4$. □

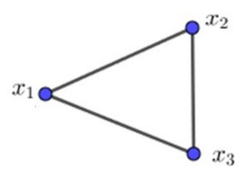
Corollary 2.7. *If $\mathcal{G}$ is a star graph, then* $girth(\Gamma(R(\mathcal{G}))) = 4$.

Proof. Since $\mathcal{G}$ is a star graph given by the pair $(V = \{x_1, \ldots, x_n\}, E = \{\{x_1, x_i\} : 2 \leqslant i \leqslant n\})$, then the unique minimal vertex covers for $\mathcal{G}$ are $\{x_1\}$ and $\{x_2, x_3, \ldots, x_n\}$ and by Proposition 2.6 we have the result. □

Proposition 2.8. *Let $\mathcal{G} = (V, E)$ be a simple graph satisfying $|V| \geqslant 3$, which is not a star graph, with at least one apex vertex. Then* $girth(\Gamma(R(\mathcal{G}))) = 3$.

Proof. Let $x_1 \in V$ be the apex vertex of $\mathcal{G}$. Since $\mathcal{G}$ is not a star, there exist $x_i, x_j \in V \setminus \{x_1\}$ such that $\{x_i, x_j\} \in E(\mathcal{G})$. Therefore, the subgraph of $\mathcal{G}$ induced by $\{x_1, x_i, x_j\}$ is a 3-cycle, and we conclude that $girth(\Gamma(R(\mathcal{G}))) = 3$. □

Example 2.9. *Let $\mathcal{G}$ be a simple graph given by the ordered pair* $(V = \{x_1, x_2, x_3\},\ E = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\})$



Then we can get a 3-cycle $x_1 \sim x_2 \sim x_3 \sim x_3 \sim x_1$. Hence $girth(\Gamma(R(\mathcal{G}))) = 3$

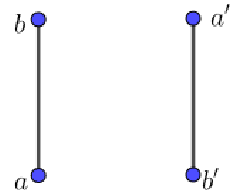
So far we have considered subgraphs induced by three vertices, in the following results we will work with subgraphs induced by say four vertices $\{a, b, a', b'\}$, but if the edges $\{a, a'\}$ and $\{b, b'\} \in E(\mathcal{G})$, one of the cases already examined in examples 2.4, 2.5 or 2.9 will be present. For this reason, it will be a hypothesis to be considered in the following.

Lemma 2.10. *Let $\mathcal{G} = (V, E)$ be a simple graph and let $\{a, b\} \in E$ such that there exist $a', b' \in V \setminus \{a, b\}$ such that $\{a, a'\}, \{b, b'\} \notin E$ and the following conditions are satisfied $\{a, b'\}; \{b, a'\} \notin E$ and $\{a', b'\} \in E$. Then* $girth(\Gamma(R(\mathcal{G}))) = 3$.

Proof. Let be the subgraph of $\mathcal{G}$ induced by $\{a, b, a', b'\}$:
Under these conditions, there exists the following 3-cycle in $\Gamma(R(\mathcal{G}))$,

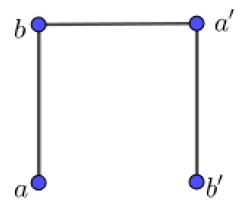
$$\overline{ab'} \sim \overline{bc} \sim \overline{aa'} \sim \overline{ab'}$$

where $c \in \{a', b'\}$. □



Lemma 2.11. *Let $\mathcal{G} = (V, E)$ be a simple graph and $\{a, b\} \in E$ such that there exist $a', b' \in V \setminus \{a, b\}$ such that $\{a, a'\}, \{b, b'\} \notin E$ and the following conditions are satisfied: $\{a, b'\} \notin E$ and $\{b, a'\}, \{a', b'\} \in E$. Then $girth(\Gamma(R(\mathcal{G}))) = 3$.*

Proof. Let be the subgraph of $\mathcal{G}$ induced by $\{a, b, a', b'\}$:



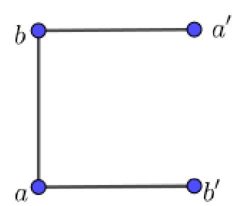
Under these conditions we get the following 3-cycle in $\Gamma(R(\mathcal{G})) = 3$,

$$\overline{aa'} \sim \overline{bb'} \sim \overline{ab'} \sim \overline{aa'}$$

whereupon we conclude the proof. □

Lemma 2.12. *Let $\mathcal{G} = (V, E)$ be a simple graph and let $\{a, b\} \in E$ such that there exist $a', b' \in V \setminus \{a, b\}$ such that $\{a, a'\}, \{b, b'\} \notin E$ and the following conditions are satisfied $\{a', b'\} \notin E$ and $\{b, a'\}, \{a, b'\} \in E$. Then $girth(\Gamma(R(\mathcal{G}))) = 3$.*

Proof. Let be the subgraph induced by $\{a, b, a', b'\}$:



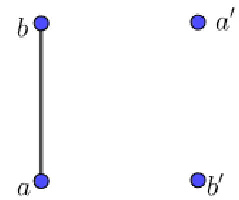
Under these conditions we get the following 3-cycle in $\Gamma(R(\mathcal{G}))$.

$$\overline{a'b'} \sim \overline{b} \sim \overline{a} \sim \overline{a'b'}$$

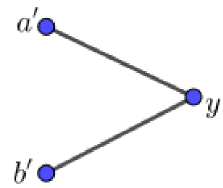
where $girth(\Gamma(R(\mathcal{G}))) = 3$. □

Lemma 2.13. *Let $\mathcal{G} = (V, E)$ be a simple graph and let $\{a, b\} \in E$ such that there exist $a', b' \in V \setminus \{a, b\}$ such that $\{a, a'\}, \{b, b'\} \notin E$ and the following conditions are satisfied $\{a, b'\}, \{b, a'\}, \{a', b'\} \notin E$. Then $girth(\Gamma(R(\mathcal{G}))) = 3$.*

Proof. The subgraph induced by $\{a, b, a', b'\}$ of $\mathcal{G}$ is:



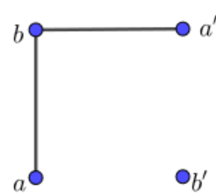
Since $\mathcal{G}$ is simple, there are not necessarily distinct vertices $y, z \in V \setminus \{a, b, a', b'\}$ such that $\{a', y\}, \{b', z\} \in E$. We first analyze the case $y = z$. Thus, the graph induced by $\{a', b', y\}$ is:



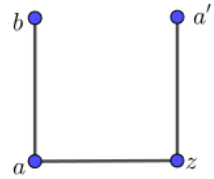
y and we will have the 3-cycle $\overline{a'b} \sim \overline{aa'} \sim \overline{y'b} \sim \overline{a'b}$ in $\Gamma(R(\mathcal{G}))$.
In the case $y \neq z$, a 3-cycle in $\Gamma(R(\mathcal{G}))$ is $\overline{a'b} \sim \overline{a'a} \sim \overline{y} \sim \overline{a'b}$. □

Lemma 2.14. *Let $\mathcal{G} = (V, E)$ be a simple graph and $\{a, b\} \in E$ such that there exist $a', b' \in V \setminus \{a, b\}$ such that $\{a, a'\}, \{b, b'\} \notin E$ and the following conditions are satisfied $\{a, b'\}, \{a', b'\} \notin E$ and $\{b, a'\} \in E$. Then $girth(\Gamma(R(\mathcal{G}))) = 3$.*

Proof. Let be the subgraph of $\mathcal{G}$ induced by $\{a, b, a', b'\}$:



Since $\mathcal{G}$ is simple, there exists $z \in V \setminus \{a, b, a', b'\}$ such that $\{b', z\} \in E$. We proceed case by case. Suppose $\{a, z\} \notin E$. So, $\overline{b} \sim \overline{ab'} \sim \overline{za} \sim \overline{b}$ is a 3-cycle in $\Gamma(R(\mathcal{G}))$. In the case $\{a, z\} \in E$, we have that the subgraph induced by $\{a, b, b', z\}$ is
Applying Lemma 2.11 to the subgraph, we have that $girth(\Gamma(R(\mathcal{G}))) = 3$. □



We say that a simple graph $\mathcal{G} = (V, E)$, satisfies property P if, given a, b, a' and $b' \in V$ distinct, $\{a, b\} \in E$ and $\{a, a'\}, \{b, b'\} \notin E$ then $\{a, b'\}, \{b, a'\}, \{a', b'\} \in E$.

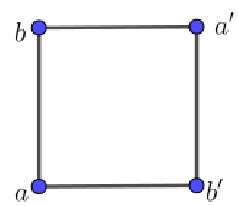
Remark 2.15. *We can see that the graphs that satisfy the property P are those graphs that do not have four vertices $\{a, b, a', b'\}$ that satisfy the conditions of some of the previous 5 lemmas.*

Definition 2.16. *A simple graph $\mathcal{G} = (V, E)$ is a **square graph** if it satisfies the property P, girth$(\mathcal{G}) \geqslant 4$ and is not a cone graph.*

Proposition 2.17. *If $\mathcal{G}$ is a simple and square graph, then $\mathcal{G}$ is connected.*

Proof. Let $\mathcal{G} = (V, E)$. Suppose that $\mathcal{G}$ is a simple graph with at least two connected components $\mathcal{C}_1 = (V_1, E_1)$ and $\mathcal{C}_2 = (V_2, E_2)$. Then there exist $\{a_1, b_1\} \in E_1$ and $\{a_2, b_2\} \in E_2$ such that $\{a_1, a_2\}, \{b_1, b_2\} \notin E$. Then $\{a_1, b_1\} \in E$ with $\{a_1, a_2\}, \{b_1, b_2\} \notin E$ and $\{a_2, b_1\}, \{a_1, b_2\} \notin E$, contradicting that $\mathcal{G}$ is a square graph. Therefore, $\mathcal{G}$ is connected. □

Example 2.18. *Consider the square graph $\mathcal{G} = (V, E)$, where $V = \{a, b, a', b'\}$ and $E = \{\{a, b\}, \{b, a'\}, \{a', b'\}, \{a, b'\}\}$.*



Note that the only minimal vertex covers are $\{a, a'\}$ and $\{b, b'\}$ by Proposition 2.6; the girth$(\Gamma(R(\mathcal{G})))$ is 4.

Consider a square graph $\mathcal{G} = (V, E)$ with $|V| \geqslant 4$. Let $\{a, b\} \in E$, which exists because $|V| \geqslant 4$ and $\mathcal{G}$ is simple. Since $\mathcal{G}$ is not a cone, there exist $a', b' \in V$ such that $\{a, a'\}, \{b, b'\} \notin E$.

Since $\mathcal{G}$ is a square graph, the property P guarantees that the subgraph induced by $\{a, b, a', b'\}$ is the same graph as the one shown in Example 2.17.

Now for any $c \in V \setminus \{a, b, a', b'\}$. It follows that if $\{a, c\} \in E$ then $\{a, b\} \in E$ and $\{a, c\}, \{b, b'\} \in E$. By property P it follows that $\{b, c\} \in E$. Analogously, if $\{b, c\} \notin E$ then $\{a, c\} \in E$. But it cannot be that $\{a, c\}, \{b, c\} \in E$ since $a \sim b \sim c \sim a$ would be a 3-cycle in $\mathcal{G}$. Moreover, from property P we have that $\{b, c\} \in E$ implies $\{b', c\} \in E$, and since $girth(\Gamma(R(\mathcal{G}))) \leqslant 4$ it follows that $\{a, c\}, \{a', c\} \notin E$. By a similar argument, if $\{a, c\} \in E$ implies that $\{a', c\} \in E$, then one would also have that $\{b, c\}, \{b', c\} \notin E$.

From the above, a square graph must be isomorphic to one of the following graphs:

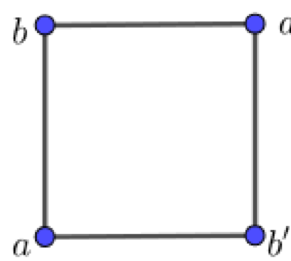


Figure 1: $\mathcal{G}_1$

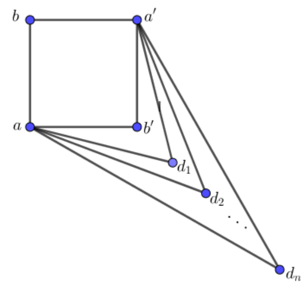


Figure 2: $\mathcal{G}_2$

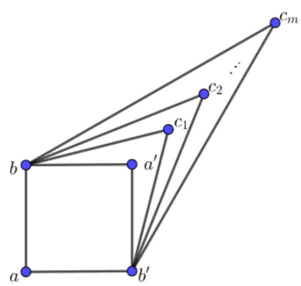


Figure 3: $\mathcal{G}_3$

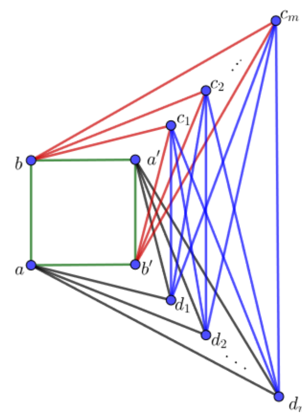


Figure 4: $\mathcal{G}_4$

Compute the prime decomposition into the four graphs by their minimal vertex covers:

$$I(\mathcal{G}_1) = \langle b, b' \rangle \cap \langle a, a' \rangle$$

$$I(\mathcal{G}_2) = \langle b, b' \rangle \cap \langle a, a', c_1, c_2, \ldots, c_m \rangle$$

$$I(\mathcal{G}_3) = \langle a, a' \rangle \cap \langle b, b', d_1, d_2, \ldots, d_n \rangle$$

$$I(\mathcal{G}_4) = \langle b, b', d_1, \cdots, d_m \rangle \cap \langle a, a', c_1, \ldots, c_m \rangle$$

And by Proposition 2.6 $girth(\Gamma(R(\mathcal{G}_i))) = 4$ for $i = \{1, 2, 3, 4\}$. We summarize this information in the following result.

Theorem 2.19. *If $\mathcal{G}$ is a square graph, then $girth(\Gamma(R(\mathcal{G}))) = 4$*

Theorem 2.20. *Let $\mathcal{G} = (V, E)$ be a simple graph with $|V| \geqslant 2$. Then*

1. *$girth(\Gamma(R(\mathcal{G}))) = 3$ or 4*
2. *$girth(\Gamma(R(\mathcal{G}))) = 4$ if $\mathcal{G}$ is a square graph or a star graph.*

As an application of Theorem 2.20 and Proposition 2.6 to graph theory, we have

Corollary 2.21. *The only simple graphs with exactly two minimal vertex covers are the star graphs and the square graphs.*

3 Diameter of $\Gamma(R(\mathcal{G}))$

In this section we will characterize the diameter of the graph $\Gamma(R(\mathcal{G}))$, as a first part of this task we will find a bound for this graph invariant. Recall that for a commutative ring R we have in general that

Theorem 3.1 ([2], Theorem 2.3)**.** *Let R be a commutative ring. Then $\Gamma(R)$ is a connected graph satisfying $0 \leqslant diam(\Gamma(R)) \leqslant 3$.*

Proof. Let $x, y \in Z(R)^*$. It suffices to show that there exists a path from x to y of length at most 3 and we have guaranteed connectedness and $0 \leqslant diam(\Gamma(R)) \leqslant 3$. Such a proof will be done by cases:

- Case 1 : $xy = 0$, i.e, $x \sim y$ and so $d(x, y) = 1$.
- Case 2 : $xy \neq 0$, then there exist $u, v \in Z(R)^*$ such that $xu = 0$ and $vy = 0$.
  - Subcase 2.1 $uv = 0$, then $x \sim u \sim v \sim y$ with $d(x, y) \leqslant 3$.
  - Subcase 2.2 $uv \neq 0$, then $x \sim uv \sim y$ with $d(x, y) = 2$.

Therefore, $d(x, y) \leqslant 3$ □

Proposition 3.2. *Let $\mathcal{G} = (V, E)$ be a simple graph. Then $2 \leqslant diam(\Gamma(R(\mathcal{G}))) \leqslant 3$*

Proof. Since for every $x \in V$ one has $1 < d(x, x^2)$. Therefore $2 \leqslant diam(\Gamma(R(G))) \leqslant 3$. □

In [6] Lucas developed a characterization for the graph of divisors of zero using the graph invariant $diam(\Gamma(R))$, this characterization is described below.

Theorem 3.3 ([6], Theorem 2.6)**.** *Let R a commutative ring.*

1. *$diam(\Gamma(R)) = 0$ if and only if R is (nonreduced and) isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/\langle X^2 \rangle$*
2. *$diam(\Gamma(R)) = 1$ if and only if $xy = 0$ $\forall x, y \in Z(R)^*$ with $x \neq y$ and $|Z(R)^*| \geqslant 2$*

3. *diam*$(\Gamma(R)) = 2$ *if and only if either*

   (a) *R is reduced with exactly two minimal primes and at least three nonzero divisors.*

   (b) *$Z(R)$ is an ideal whose square is not $\{0\}$ and each pair of distinct zero divisors has a nonzero annihilator.*

4. *diam*$(\Gamma(R)) = 3$ *if and only if there are zero divisors $a \neq b$ such that $(0{:}\langle a, b\rangle) = \{0\}$ and either*

   (a) *R is a reduced ring with more that two minimal primes.*

   (b) *R is nonreduced.*

Proposition 3.4. *Let $\mathcal{G}$ be a simple graph with exactly two minimal vertex covers. Then $\Gamma(R(\mathcal{G}))$ is a complete bipartite graph.*

Proof. Let C_1, C_2 be the minimal vertex covers of $\mathcal{G}$ and V the set of vertices $\{x_1, x_2, \ldots, x_n\}$. We can assume without loss of generality that $C_1 = \{x_1, \ldots, x_s\}$ and $C_2 = \{x_{s+1}, \ldots, x_n\}$ since they must be irrelevant. So, $Z(R(\mathcal{G})) = \langle C_1\rangle \cup \langle C_2\rangle$ and $\Gamma(R(\mathcal{G})) = K_{\langle C_1\rangle, \langle C_2\rangle}$ □

Corollary 3.5. *If $\mathcal{G}$ has exactly two minimal vertex covers, then diam*$(\Gamma(R(\mathcal{G}))) = 2$

Recalling that $\mathcal{G} = (V, E)$ is a simple graph, it follows from Proposition 3.2 that $2 \leqslant$ *diam*$(\Gamma(R(\mathcal{G}))) \leqslant 3$. Furthermore, $R(\mathcal{G}) = K[x_1, \ldots, x_n]/I(\mathcal{G})$ with $I(\mathcal{G}) = \bigcap_{i=1}^{m}\langle C_i\rangle$, where $C_1, \ldots, C_m$ are the minimal vertex covers. Moreover,

$$nil(R(\mathcal{G})) = \sqrt{\overline{0}} = \sqrt{I(\mathcal{G})} = \sqrt{\bigcap_{i=1}^{m}\langle C_i\rangle} = \bigcap_{i=1}^{m}\sqrt{\langle C_i\rangle} = \bigcap_{i=1}^{m}\langle C_i\rangle = \overline{0}$$

where we can see that $R(\mathcal{G})$ is reduced. Considering Theorem 3.3 and the fact that $R(\mathcal{G})$ is reduced. We can state the following.

Theorem 3.6. *Let $\mathcal{G} = (V,\ E)$ be a simple graph. Then*

1. *diam*$(\Gamma(R(\mathcal{G}))) = 2$ *if and only if $\mathcal{G}$ has exactly two minimal vertex covers*

2. *diam*$(\Gamma(R(\mathcal{G}))) = 3$ *if and only if there exist $P(\overline{X}), Q(\overline{X}) \in Z(R(\mathcal{G}))$ such that $P(\overline{X}) \neq Q(\overline{X})$ and $(\overline{0}{:}\langle P(\overline{X})Q(\overline{X})\rangle) = \{\overline{0}\}$ and $\mathcal{G}$ has more than two minimal vertex covers.*

4 Clique number: $\omega(\Gamma(R(\mathcal{G})))$

In this section we study the graphic clique number invariant $\omega(\Gamma(R(\mathcal{G})))$, and we characterize the family of graphs of zero divisors of the ring of edges of a graph $\mathcal{G}$, based on the number of minimal vertex covers of the graph. Let $\mathcal{G}$ be a graph which has only two minimal vertex covers. By Proposition 3.4, $\Gamma(R(\mathcal{G})) = K_{\langle C_1\rangle, \langle C_2\rangle}$, furthermore, it is known that given a complete k-partite graph, its clique number is k. Thus, it is concluded that $\omega(\Gamma(R(\mathcal{G}))) = 2$.

For the case where graph $\mathcal{G}$ has three minimal vertex covers C_1, C_2, C_3. Remember that in the ring of edges of a simple graph $R(\mathcal{G})$ then $0 = I(\mathcal{G}) = \bigcap_{j=1}^{3} \langle C_j \rangle$ in this way the set of divisors of zero is:

$$Z(R(\mathcal{G}))^* = \left\{ z \in \bigcap_{j \in \{1,2,3\} \setminus B} \langle C_j \rangle \setminus \bigcup_{i \in B} \langle C_i \rangle : \emptyset \neq B \subsetneq \{1,2,3\} \right\} \tag{1}$$

for $I = \{1,2,3\}$; seen as are the elements that belong to $Z(R(\mathcal{G}))^*$, we will define the sets $A_{I \setminus B}$ with $B \in P(I) \setminus \{\emptyset, I\}$ as follows:

$$A_{I \setminus B} := \bigcap_{j \in I \setminus B} \langle C_j \rangle \setminus \bigcup_{i \in B} \langle C_i \rangle \tag{2}$$

Here $P(I)$ symbolizes the power set of the set I. We now list all possible sets $A_{I \setminus B}$ for $I = \{1,2,3\}$

$$A_{I \setminus \{1\}} = \langle C_2 \rangle \cap \langle C_3 \rangle \setminus \langle C_1 \rangle$$
$$A_{I \setminus \{2\}} = \langle C_1 \rangle \cap \langle C_3 \rangle \setminus \langle C_2 \rangle$$
$$A_{I \setminus \{3\}} = \langle C_1 \rangle \cap \langle C_2 \rangle \setminus \langle C_3 \rangle$$
$$A_{I \setminus \{2,3\}} = \langle C_1 \rangle \setminus \langle C_2 \rangle \cup \langle C_3 \rangle$$
$$A_{I \setminus \{1,3\}} = \langle C_2 \rangle \setminus \langle C_1 \rangle \cup \langle C_3 \rangle$$
$$A_{I \setminus \{1,2\}} = \langle C_3 \rangle \setminus \langle C_1 \rangle \cup \langle C_2 \rangle$$

Remark 4.1. *The sets $A_{I \setminus \{p\}}$ for $p \in \{1,2,3\}$ are different from emptiness. Since $A_{I \setminus \{p\}} = A_{\{i,j\}}$ for $i \neq j$. If $A_{\{i,j\}} = \emptyset$, $\langle C_s \rangle \subset (\langle C_i \rangle \cup \langle C_j \rangle)$ by the Prime Avoidance Theorem (Theorem 3.61) en [7], $\langle C_s \rangle \subset \langle C_i \rangle$ o $\langle C_j \rangle \subset \langle C_s \rangle$, which cannot happen because they are covered by minimal vertices.*

Then for the case of three covers by minimal vertices we have that the graph $\Gamma(R(\mathcal{G}))$ would be:

$$K_{A_{\{2,3\}}, A_{\{1,3\}}, A_{\{1,2\}}} * K_{A_{\{2,3\}}, A_{\{1\}}} * K_{A_{\{1,3\}}, A_{\{2\}}} * K_{A_{\{1,2\}}, A_{\{3\}}} \tag{3}$$

Remark 4.2. *From (1) and (2) which are valid for the case where $\mathcal{G}$ has three covers, we can generalize the description of $Z(R(\mathcal{G}))^*$ for a graph $\mathcal{G}$, which has n minimal vertex covers. Let $C_1, C_2, \ldots, C_n$ be the minimal vertex covers for the graph $\mathcal{G}$ then $I = \{1,2,\ldots,n\}$, as follows:*

$$Z(R(\mathcal{G}))^* = \bigsqcup_{\emptyset \neq B \subsetneq I} A_{I \setminus B}. \tag{4}$$

Corollary 4.3. *Let $\mathcal{G}$ be a simple graph with three minimal vertex covers, then $\omega(\Gamma(R(\mathcal{G}))) = 3$*

Proof. Because of Observation 4.1 and the fact that $\Gamma(R(\mathcal{G}))$ is a graph like the one shown in (3), $\omega(\Gamma(R(\mathcal{G}))) = 3$. □

Lemma 4.4. *Let $B_1, B_2, \ldots, B_k \subsetneq I = \{1, \ldots, n\}$ be such that:*

1. *$\forall\ i \neq j$, $B_i \cup B_j = I$*
2. *$\forall\ i \neq j$, $B_i \nsubseteq B_j$.*

Then $k \leqslant n$.

Lemma 4.5. *Let $C_1, \ldots, C_n$ be the minimal vertex covers of the graph $\mathcal{G}$. Then $\omega(\Gamma(R(\mathcal{G}))) = n$.*

Proof. Remember that for $I = \{1, 2, \ldots, n\}$ the sets $A_{I\setminus B}$ were defined in (2), furthermore we know that the set of divisors of zero can be seen as (4). Before the test let's make some observations:

- Let $\emptyset \neq B_0 \in \mathbb{P}(I) \setminus I$, $x \in A_{I\setminus B_0}$ and $y \in Z(R)^*$. Then $xy = 0$ if and only if $y \in A_{I\setminus B_0^C}$
- Let $B_1, B_2 \in \mathbb{P}(I) \setminus I$ be not empty. Then $A_{I\setminus B_1} \subseteq A_{I\setminus B_2}$ if and only if $B_1 \subseteq B_2$.

Like in case $n = 3$ graph $\Gamma(R(\mathcal{G}))$ is made up of complete k-partite graphs of the form $K_{A_{I\setminus B_1}, A_{I\setminus B_2}, \ldots, A_{I\setminus B_k}}$ where $B_i \nsubseteq B_j$ and $B_i \cup B_j = I$ for any pair $i, j \in \{1, 2, \ldots, k\}$. And the largest k with said characteristics is $k = n$ with $B_i = i$ for $1 \leq i \leq n$. Therefore, $\omega(\Gamma(R(\mathcal{G}))) = n$. □

5 Chromatic number: $\chi(\Gamma(R(\mathcal{G})))$

In this section we now characterize the chromatic number of the graphs of zero divisors of the edge ring for a simple graph, as it was said before, we must find a coloring for the vertices, where two adjacent vertices have different colors.

Recall that a vertex coloring is a function ϕ defined as follows, $\phi : V(\mathcal{G}) \to \mathbb{N}$ so that if $x \sim y$ then $\phi(x) \neq \phi(y)$, it is evident that if we assign a different natural number to each of the vertices, it is easy to satisfy this condition, but for our purposes, we seek to find the smallest number k, in such a way that this property continues to be fulfilled, said $k \in N$ defining the function $\phi : V(\mathcal{G}) \to \{1, \ldots, k\}$ so that if $x \sim y$ then $\phi(x) \neq \phi(y)$ is the chromatic number of $\mathcal{G}$, in symbols $\chi(\mathcal{G}) = k$.

With the intention of finding a coloring for the graph of zero divisors of the ring of edges, let's explore some examples of graphs $\mathcal{G}$ to find the procedure that will determine the minimum coloration of the vertices of $\Gamma(\mathcal{G})$. The examples that we will study will be the cases where the graph has two, three and four minimal vertex covers.

Example 5.1. ***Coloring for a graph with two covers by minimal vertices***
We can consider a graph $\mathcal{G}$ that possess the covers C_1, C_2, then $I = \{1, 2\}$ then the sets described by expression (2) in section 4, are as follows:

$$A_{I\setminus\{1\}} = \langle C_2 \rangle \setminus \langle C_1 \rangle$$

$$A_{I\setminus\{2\}} = \langle C_1\rangle \setminus \langle C_2\rangle$$

Defining the function ϕ as follows

$$\begin{array}{ccc} \varphi : Z(R)^* & \longrightarrow & I \\ A_{I\setminus\{1\}} & \mapsto & 1 \\ A_{I\setminus\{2\}} & \mapsto & 2 \end{array}$$

Since the function ϕ is a coloring for the graph $\Gamma(R(\mathcal{G}))$ it follows then that if $\mathcal{G}$ has two minimal vertex covers then $\chi(\Gamma(R(\mathcal{G}))) = 2$.

In Example 5.1 we found how to determine a coloring for a graph $\mathcal{G}$ with two covers by minimal vertices. Now let's check the behavior of a graph $\mathcal{G}$ which has three minimal vertex covers C_1, C_2 and C_3.

Given $\mathcal{G}$ a simple graph with minimal vertex covers C_1, C_2, C_3 in section 4 it was possible to see that

$$Z(R(\mathcal{G}))^* = A_{\{1,2\}} \cup A_{\{2,3\}} \cup A_{\{1,3\}} \cup A_{\{1\}} \cup A_{\{2\}} \cup A_{\{3\}}$$

and even more

$$\Gamma(R(\mathcal{G})) = K_{A_{\{1,2\}},A_{\{1,3\}},A_{\{2,3\}}} * K_{A_{\{1,2\}},A_{\{3\}}} * K_{A_{\{1,3\}},A_{\{2\}}} * K_{A_{\{2,3\}},A_{\{1\}}}$$

within the results obtained in the previous section we can remember that $\omega(\Gamma(R(\mathcal{G}))) = 3$ when $\mathcal{G}$ has three minimal vertex covers. From a known fact in the theory of graphs it is known that given a graph $\mathcal{G}$, we have that $\omega(\mathcal{G}) \leqslant \chi(\mathcal{G})$.

So in a graph $\mathcal{G}$, which has three covers by minimal vertices C_1, C_2 and C_3 we have that $\chi(\Gamma(R(\mathcal{G}))) \geqslant 3$.

Note that for each pair $\{i, j\}$ we have that $A_{\{i,j\}} \neq \emptyset$. Since there exist $x \in C_i \setminus C_s$ and $y \in C_j \setminus C_s$ so that $xy \in A_{\{i,j\}}$. However $A_{\{1\}}, A_{\{2\}}, A_{\{3\}}$ all or some of them can be empty. No matter which $A_{\{i\}}$ we have that $\chi(\Gamma(R(\mathcal{G}))) = 3$. To corroborate this fact, we will give the example of a coloring function ϕ.

Example 5.2. ***Coloring for a graph with three covers by minimal vertices***
Let us now consider a graph $\mathcal{G}$ which has three minimal vertex covers C_1, C_2, C_3, so that $I = \{1, 2, 3\}$ then the sets $A_{I\setminus B}$ are as follows:

$$A_{I\setminus\{1\}} = \langle C_2\rangle \cap \langle C_3\rangle \setminus \langle C_1\rangle$$

$$A_{I\setminus\{2\}} = \langle C_1\rangle \cap \langle C_3\rangle \setminus \langle C_2\rangle$$

$$A_{I\setminus\{3\}} = \langle C_1\rangle \cap \langle C_2\rangle \setminus \langle C_3\rangle$$

$$A_{I\setminus\{2,3\}} = \langle C_1\rangle \setminus \langle C_2\rangle \cup \langle C_3\rangle$$

$$A_{I\setminus\{1,3\}} = \langle C_2\rangle \setminus \langle C_1\rangle \cup \langle C_3\rangle$$

$$A_{I\setminus\{1,2\}} = \langle C_3\rangle \setminus \langle C_1\rangle \cup \langle C_2\rangle$$

We can assume without loss of generality that each element of the set $A_{I\setminus\{i\}}$ receives color i, where $i \in \{1,2,3\}$. Given that $A_{I\setminus\{1\}}$ it is already color 1 then $A_{I\setminus\{2,3\}}$ can be of color 2 or 3, following the same reasoning, from the fact of having $A_{I\setminus\{2\}}$ with color 2 then $A_{I\setminus\{1,3\}}$ can be color 1 or 3, when choosing color 3 for the elements of the set $A_{I\setminus\{3\}}$ then $A_{I\setminus\{1,2\}}$ can be color 1 or 2. This information can be summarized in the following table: This table contains all possible colorings for the graph $\Gamma(R(\mathcal{G}))$ after assigning to set $A_{I\setminus\{i\}}$

Sets	Possible colorings
$A_{I\setminus\{1\}}$	1
$A_{I\setminus\{2\}}$	2
$A_{I\setminus\{3\}}$	3
$A_{I\setminus\{2,3\}}$	2 or 3
$A_{I\setminus\{1,3\}}$	1 or 3
$A_{I\setminus\{1,2\}}$	1 or 2

Table 1: Possible colorings for $\Gamma(R(\mathcal{G}))$,
if $\mathcal{G}$ has 3 covers by minimal vertices.

color i. We choose one of them and define the function

$$\begin{array}{rcl} \varphi : Z(R)^* & \longrightarrow & I \\ A_{I\setminus\{1\}} & \mapsto & 1 \\ A_{I\setminus\{2\}} & \mapsto & 2 \\ A_{I\setminus\{3\}} & \mapsto & 3 \\ A_{I\setminus\{2,3\}} & \mapsto & 2 \\ A_{I\setminus\{1,3\}} & \mapsto & 1 \\ A_{I\setminus\{1,2\}} & \mapsto & 1 \end{array}$$

Therefore, if $\mathcal{G}$ has three minimal vertex covers then $\chi(\Gamma(R(\mathcal{G}))) = 3$.

In the case where the graph $\mathcal{G}$ have four covers for minimal vertices, the process to be able to assign a coloring to the vertices belonging to the different sets $A_{I\setminus B}$ for $B \notin \{\emptyset, I\}$ with $B \subset I$ of the graph $\Gamma(R(\mathcal{G}))$, it is possible that with arguments similar to those used when the graph $\mathcal{G}$ has three minimal vertex covers, are useful, which will be seen in the following example.

Example 5.3. ***Coloring for a graph with four covers by minimal vertices***
For $n = 4$ we can consider the covers C_1, C_2, C_3 and C_4, so $I = \{1,2,3,4\}$ then the sets $A_{I\setminus B}$ are as follows:

$$A_{I\setminus\{1\}} = \langle C_2\rangle \cap \langle C_3\rangle \cap \langle C_4\rangle \setminus \langle C_1\rangle$$

$$A_{I\setminus\{2\}} = \langle C_1\rangle \cap \langle C_3\rangle \cap \langle C_4\rangle \setminus \langle C_2\rangle$$

$$A_{I\setminus\{3\}} = \langle C_1\rangle \cap \langle C_2\rangle \cap \langle C_4\rangle \setminus \langle C_3\rangle$$

$$A_{I\setminus\{4\}} = \langle C_1\rangle \cap \langle C_2\rangle \cap \langle C_3\rangle \setminus \langle C_4\rangle$$

$$A_{I\setminus\{1,2\}} = \langle C_3\rangle \cap \langle C_4\rangle \setminus \langle C_1\rangle \cup \langle C_2\rangle$$

$$A_{I\setminus\{1,3\}} = \langle C_2\rangle \cap \langle C_4\rangle \setminus \langle C_1\rangle \cup \langle C_3\rangle$$

$$A_{I\setminus\{1,4\}} = \langle C_2\rangle \cap \langle C_3\rangle \setminus \langle C_1\rangle \cup \langle C_4\rangle$$

$$A_{I\setminus\{2,3\}} = \langle C_1\rangle \cap \langle C_4\rangle \setminus \langle C_2\rangle \cup \langle C_3\rangle$$

$$A_{I\setminus\{2,4\}} = \langle C_1\rangle \cap \langle C_3\rangle \setminus \langle C_2\rangle \cup \langle C_4\rangle$$

$$A_{I\setminus\{3,4\}} = \langle C_1\rangle \cap \langle C_2\rangle \setminus \langle C_3\rangle \cup \langle C_4\rangle$$

$$A_{I\setminus\{1,2,3\}} = \langle C_4\rangle \setminus \langle C_1\rangle \cup \langle C_2\rangle \cup \langle C_3\rangle$$

$$A_{I\setminus\{1,2,4\}} = \langle C_3\rangle \setminus \langle C_1\rangle \cup \langle C_2\rangle \cup \langle C_4\rangle$$

$$A_{I\setminus\{1,3,4\}} = \langle C_2\rangle \setminus \langle C_1\rangle \cup \langle C_3\rangle \cup \langle C_4\rangle$$

$$A_{I\setminus\{2,3,4\}} = \langle C_1\rangle \setminus \langle C_2\rangle \cup \langle C_3\rangle \cup \langle C_4\rangle$$

With an analogous procedure to the one carried out in Example 5.2, we obtain the following table: This table contains all the colorings for the graph $\Gamma(R(\mathcal{G}))$ *assigning to set* $A_{I\setminus\{i\}}$ *the*

Sets	Possible colorings
$A_{I\setminus\{1\}}$	1
$A_{I\setminus\{2\}}$	2
$A_{I\setminus\{3\}}$	3
$A_{I\setminus\{4\}}$	4
$A_{I\setminus\{1,2\}}$	1 or 2
$A_{I\setminus\{1,3\}}$	1 or 3
$A_{I\setminus\{1,4\}}$	1 or 4
$A_{I\setminus\{2,3\}}$	2 or 3
$A_{I\setminus\{2,4\}}$	2 or 4
$A_{I\setminus\{3,4\}}$	3 or 4
$A_{I\setminus\{1,2,3\}}$	1 or 2 or 3
$A_{I\setminus\{1,2,4\}}$	1 or 2 or 4
$A_{I\setminus\{1,3,4\}}$	1 or 3 or 4
$A_{I\setminus\{2,3,4\}}$	2 or 3 or 4

Table 2: Possible colorings for $\Gamma(R(\mathcal{G}))$,
if $\mathcal{G}$ has 4 covers by minimal vertices.

color i. We choose one of them and define the function

$$\begin{array}{rcl} \varphi : Z(R)^* & \longrightarrow & I \\ A_{I\setminus\{1\}} & \mapsto & 1 \\ A_{I\setminus\{2\}} & \mapsto & 2 \\ A_{I\setminus\{3\}} & \mapsto & 3 \\ A_{I\setminus\{4\}} & \mapsto & 4 \\ A_{I\setminus\{1,2\}} & \mapsto & 1 \\ A_{I\setminus\{1,3\}} & \mapsto & 1 \\ A_{I\setminus\{1,4\}} & \mapsto & 1 \\ A_{I\setminus\{2,3\}} & \mapsto & 2 \\ A_{I\setminus\{2,4\}} & \mapsto & 2 \\ A_{I\setminus\{3,4\}} & \mapsto & 3 \\ A_{I\setminus\{1,2,3\}} & \mapsto & 1 \\ A_{I\setminus\{1,2,4\}} & \mapsto & 1 \\ A_{I\setminus\{1,3,4\}} & \mapsto & 1 \\ A_{I\setminus\{2,3,4\}} & \mapsto & 2 \end{array}$$

From Examples 5.1, 5.2 and 5.3 you can see the algorithm with which we will find one of the possible colorings for the graph $\Gamma(R(\mathcal{G}))$ according to the number of covers by minimal vertices, that the graph $\mathcal{G}$ has, now to proceed to a proof of the general case when $\mathcal{G}$ possess n covered by minimal vertices, we will first prove a lemma, which guarantees us when a function ϕ defined of $Z(R(\mathcal{G}))^*$ to $I = \{1, \cdots, n\}$ is a coloring.

Lemma 5.4. *Let $\varphi : Z(R(\mathcal{G}))^* \longrightarrow I$ be a function. Then φ is a vertex coloring for $\Gamma(R(\mathcal{G}))$ if and only if for all $B, C \subset I$, $B \neq \emptyset \neq C$, $B \cap C = \emptyset$*

$$\varphi(A_{I\setminus B}) \cap \varphi(A_{I\setminus C}) = \emptyset.$$

Proof. First, suppose that φ is a vertex coloring for the graph and let $B, C \subset I$ with $B \neq \emptyset \neq C$ and $B \cap C = \emptyset$. So for each pair $x \in A_{I\setminus B}$, $y \in A_{I\setminus C}$ we have that $xy = 0$. Therefore, $\{x, y\} \in E(\Gamma(R(\mathcal{G})))$ and since φ is a coloring then $\varphi(x) \neq \varphi(y)$. We conclude that $\varphi(A_{I\setminus B}) \cap \varphi(A_{I\setminus C}) = \emptyset$.

On the other hand, let $x, y \in Z(R(\mathcal{G}))^*$ so that $x, y \in E(R(\mathcal{G}))$, meaning $xy = 0$. For $Z(R(\mathcal{G}))^* = \bigsqcup_{\emptyset \neq B \subseteq I} A_{I\setminus B}$, there exist $B, C \subset I$ with $B \neq \emptyset \neq C$, $x \in A_{I\setminus B}$ and $y \in A_{I\setminus C}$. Since $xy = 0$ then $B \cap C = \emptyset$. By hypothesis $\varphi(A_{I\setminus B}) \cap \varphi(A_{I\setminus C}) = \emptyset$ concluding that $\varphi(x) \neq \varphi(y)$, meaning, φ is a vertex coloring for $\Gamma(R(\mathcal{G}))$. □

Once this is done, we have all the necessary elements to be able to demonstrate what the chromatic number of $\Gamma(R(\mathcal{G}))$ when $\mathcal{G}$ has n minimal vertex covers is.

Theorem 5.5. *Given a simple graph $\mathcal{G}$, with $C_1, C_2, \ldots, C_n$ the minimal vertex covers and considering $\Gamma(R(\mathcal{G}))$. Then $\chi(\Gamma(R(\mathcal{G}))) = n$*

Proof. The proof will be constructive by assigning a color to each set $A_{I\setminus B}$, proceeding inductively on $|B|$. For each $i \in I$ we define $\varphi(A_{I\setminus\{i\}}) = \{i\}$, which we can do given that $I = \{1, 2, ..., n\}$. Now for each pair $\{r, s\} \in I$ we define $\varphi(A_{I\setminus\{r,s\}})$ complying with:

1. $\varphi(A_{I\setminus\{r,s\}}) = \{r\}$ or $\varphi(A_{I\setminus\{r,s\}}) = \{s\}$
2. $\varphi(A_{I\setminus\{r,s\}}) \cap \varphi(A_{I\setminus\{p,q\}}) = \emptyset$ for $\{r,s\} \cap \{p,q\} = \emptyset$

Thus proceeding inductively k times satisfying that:

$$\forall B, C \subsetneq I,\ B, C \neq \emptyset,\ B \cap C = \emptyset,\ |B|, |C| \leqslant k+1 \tag{5}$$

it is true that $\varphi(A_{I\setminus B}) \cap \varphi(A_{I\setminus C}) = \emptyset$. Now we guarantee that you can proceed to the step $k+1$ complying with the property 5. Let $B, C \subsetneq I$, $B, C \neq \emptyset$, $B \cap C = \emptyset$, $|B|, |C| \leqslant k+1$. Consider $B = \{i_1, \ldots, i_{k+1}\}$ and $C = \{j_1, \ldots, j_m\}$ with $m \leqslant k+1$, given that $|B|, |C| < k+1$ is guaranteed by 5. For $B^{'} = B \setminus \{i_{k+1}\}$ and $C^{'} = C \setminus \{j_m\}$, so $\varphi(A_{I\setminus B'}) = \{i_1\}$ or $\{i_2\}$ or $\cdots$ or $\{i_k\}$ and $\varphi(A_{I\setminus C'}) = \{j_1\}$ or $\ldots$ or $\{j_{m-1}\}$ and we define $\varphi(A_{I\setminus B}) = \varphi(A_{I\setminus B'})$ and in case $m = k+1$ we define $\varphi(A_{I\setminus C}) = \varphi(A_{I\setminus C'})$ and since $B^{'} \cap C^{'} = \emptyset$ then $\varphi(A_{I\setminus B}) = \varphi(A_{I\setminus C}) = \emptyset$. Furthermore, by Lemma 5.4 we have that φ is a vertex coloring for the graph $\Gamma(R(\mathcal{G}))$. Therefore $\chi(\Gamma(R(\mathcal{G}))) = n$ □

Now with the combination of Lemma 4.5 and Theorem 5.5 we can state the following theorem.

Theorem 5.6. *The graph $\Gamma(R(\mathcal{G}))$ is a weakly perfect graph with $\omega(\Gamma(R(\mathcal{G})))$ and $\chi(\Gamma(R(\mathcal{G})))$ equal to n, for any graph $\mathcal{G}$ where n is the number of minimal vertex covers of the graph $\mathcal{G}$.*

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