Bull. Math. Soc. Sci. Math. Roumanie
Tome 69 (117), No. 1, 2026, 15–42

Algebra environments III. Geometric structures on smooth vector bundles

by
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Abstract

The article analyzes geometric structures on smooth inner product vector bundles determined by geometric algebras, a class that includes matrix, group with two–cocycle, and Euclidean Clifford algebras. Part one of the article introduces, as requisites, the concepts of algebra environments, structure manifolds, Zariski tangent spaces, and derivations on algebra environments. Part two provides the definitions of geometric algebras and their associated Hom environments. The main result identifies the structure manifolds of Hom environments with spaces of algebra homomorphisms. Part three develops an algebraic approach to the study of geometric structures on smooth vector bundles, and concludes with characterizations of derivations and linear connections that preserve prescribed structures.

Key Words: Algebra environments, structure manifolds, Zariski tangent spaces, smooth vector bundles, geometric structures, linear connections.
2020 Mathematics Subject Classification: 14A22, 53B05, 53B15.

1 Introduction

Suppose $\mathcal{E} = (E, \pi, M)$ is a smooth inner product vector bundle over a smooth manifold M with total space E and projection $\pi : E \to M$. Let $\mathfrak{A}(M, E)$ be the algebra of smooth endomorphisms of bundle $\mathcal{E}$, let $\mathfrak{S}$ be an involutive algebra, and assume that $\Phi : \mathfrak{S} \to \mathfrak{A}(M, E)$ is an involutive algebra homomorphism, called an $\mathfrak{S}$–structure on $\mathcal{E}$. The structure Φ makes each fiber $E_x = \pi^{-1}(x)$, $x \in M$, of bundle $\mathcal{E}$ an $\mathfrak{S}$–module. A linear connection ∇ on $\mathcal{E}$ is compatible with Φ provided each ∇–parallel translation mapping $P(\nabla, c) : E_{c(0)} \to E_{c(1)}$ along a smooth curve $c : [0, 1] \to M$ is an $\mathfrak{S}$–module isomorphism. The objects of interest are the derivations and linear connections on such bundles compatible with the underlying structures.

Our goal is to construct all such linear connections by assuming that $\mathfrak{S}$ is a geometric algebra. This specific issue is related to and augments the research project developed in Martin [39, 40]. We should mention that the term geometric algebra usually refers to Euclidean Clifford algebras. The concept that we will be using defines a larger class that includes arbitrary matrix and group with two–cocycle algebras.

To make the article self–contained, Section 2 recalls the definitions of algebra environments, associated structure manifolds, Zariski tangent spaces to structure manifolds, and derivations on algebra enviroments, with descriptions of the tools used in their investigation. As additional requisites, the section includes proofs of technical results that indicate relevant properties of algebra environments.

Section 3 introduces the class of geometric algebras and the concept of Hom environments associated with a geometric algebra $\mathfrak{G}$ and an algebra $\mathfrak{A}$. The main result identifies the structure manifolds of such environments with spaces of algebra homomorphisms from $\mathfrak{G}$ to $\mathfrak{A}$.

Section 4 investigates the spaces of derivations and linear connections on vector bundles $\mathcal{E} = (E, \pi, M)$ preserving an $\mathfrak{G}$–structure $\Phi : \mathfrak{G} \to \mathfrak{A}(M, E)$. The final results characterize both spaces using an involution operator defined in terms of the prescribed structure.

Section 5 provides comments and brief descriptions of lines of research in differential geometry, Clifford analysis, harmonic analysis, and multivariable operator theory related to the theme of our article, with references.

2 Algebra Environments — Basic Concepts

This section outlines a general setting centered on the concepts of algebra environments and their associated structure manifolds and Zariski tangent spaces, which provides part of the requisites used in addressing the main theme of our article. The term algebra without other specifications refers to associative and distributive real algebras.

2.1 Algebra Environments

Definition 2.1. *An algebra environment* $(\mathfrak{E}, \Pi, \mathfrak{A})$ *consists of a total algebra* $\mathfrak{E}$*, a base algebra* $\mathfrak{A}$*, and a mapping* $\Pi : \mathfrak{E} \to \mathfrak{A}$ *called the environment projection, subject to the following requirements,*

(i) *the base algebra* $\mathfrak{A}$ *is unital with unit* $1_{\mathfrak{A}}$*,*

(ii) *the total algebra* $\mathfrak{E}$ *is a unital* $\mathfrak{A}$*–bimodule,*

(iii) $\Pi : \mathfrak{E} \to \mathfrak{A}$ *is an* $\mathfrak{A}$*–bilinear mapping.*

Whenever convenient, we refer to $(\mathfrak{E}, \Pi, \mathfrak{A})$ as environment $\mathfrak{E}$. The elements of $\mathfrak{A}$ are denoted by $a, b, \dots, x, y, \dots$, and the elements of $\mathfrak{E}$ by $\alpha, \beta, \dots, \varphi, \psi, \dots$. The products of $x, y \in \mathfrak{A}$, or $\varphi, \psi \in \mathfrak{E}$, are expressed as xy and $\varphi \times \psi$. The left and right products of $a \in \mathfrak{A}$ and $\varphi \in \mathfrak{E}$ are denoted by $a \cdot \varphi$ and $\varphi \cdot a$. Requirements (ii) and (iii) in Definition 2.1 include the specific rules

$$a \cdot (\varphi \times \psi) = (a \cdot \varphi) \times \psi, \;\; (\varphi \times \psi) \cdot a = \varphi \times (\psi \cdot a), \;\; (\varphi \cdot a) \times \psi = \varphi \times (a \cdot \psi),$$

$$1_{\mathfrak{A}} \cdot \varphi = \varphi \cdot 1_{\mathfrak{A}} = \varphi, \quad \Pi(a \cdot \varphi) = a\, \Pi(\varphi), \quad \Pi(\varphi \cdot a) = \Pi(\varphi)\, a,$$

for any $a \in \mathfrak{A}$ and φ, $\psi \in \mathfrak{E}$.

Definition 2.2. *An environment* $(\mathfrak{E}, \Pi, \mathfrak{A})$ *is involutive provided* $\mathfrak{E}$ *and* $\mathfrak{A}$ *are involutive algebras with the involution operations denoted by* $*$*, such that*

$$(a \cdot \varphi)^* = \varphi^* \cdot a^*, \quad (\varphi \cdot a)^* = a^* \cdot \varphi^*, \quad a \in \mathfrak{A}, \;\; \varphi \in \mathfrak{E},$$

$$\Pi(\varphi^*) = \Pi(\varphi)^*, \quad \varphi \in \mathfrak{E}.$$

Definition 2.3. *An environment homomorphism from* $(\mathfrak{E}, \Pi_{\mathfrak{E}}, \mathfrak{A})$ *to* $(\mathfrak{F}, \Pi_{\mathfrak{F}}, \mathfrak{B})$ *is a pair* (Λ, Λ_0) *where* $\Lambda : \mathfrak{E} \to \mathfrak{F}$ *and* $\Lambda_0 : \mathfrak{A} \to \mathfrak{B}$ *are algebra homomorphisms compatible with the bimodule structures, i.e.,*

$$\Lambda(a \cdot \varphi) = \Lambda_0(a) \cdot \Lambda(\varphi), \quad \Lambda(\varphi \cdot a) = \Lambda(\varphi) \cdot \Lambda_0(a), \;\; a \in \mathfrak{A}, \;\; \varphi \in \mathfrak{E},$$

satisfying the additional properties

$$\Lambda_0(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}, \quad \Lambda_0 \circ \Pi_{\mathfrak{E}} = \Pi_{\mathfrak{F}} \circ \Lambda.$$

If $(\mathfrak{E}, \Pi_{\mathfrak{E}}, \mathfrak{A})$ *and* $(\mathfrak{F}, \Pi_{\mathfrak{F}}, \mathfrak{B})$ *are involutive,* $(\Lambda, \Lambda_0) : (\mathfrak{E}, \Pi_{\mathfrak{E}}, \mathfrak{A}) \to (\mathfrak{F}, \Pi_{\mathfrak{F}}, \mathfrak{B})$ *is an involutive environment homomorphism provided* $\Lambda : \mathfrak{E} \to \mathfrak{F}$ *and* $\Lambda_0 : \mathfrak{A} \to \mathfrak{B}$ *are involutive algebra homomorphisms.*

The category of algebra environments is closed under two natural operations.

Definition 2.4. *The direct sum and tensor product of environments* $(\mathfrak{E}, \Pi_{\mathfrak{E}}, \mathfrak{A})$ *and* $(\mathfrak{F}, \Pi_{\mathfrak{F}}, \mathfrak{B})$ *are defined by*

(i) $(\mathfrak{E}, \Pi_{\mathfrak{E}}, \mathfrak{A}) \oplus (\mathfrak{F}, \Pi_{\mathfrak{F}}, \mathfrak{B}) = (\mathfrak{E} \oplus \mathfrak{F}, \Pi_{\mathfrak{E}} \oplus \Pi_{\mathfrak{F}}, \mathfrak{A} \oplus \mathfrak{B})$,

(ii) $(\mathfrak{E}, \Pi_{\mathfrak{E}}, \mathfrak{A}) \otimes (\mathfrak{F}, \Pi_{\mathfrak{F}}, \mathfrak{B}) = (\mathfrak{E} \otimes \mathfrak{F}, \Pi_{\mathfrak{E}} \otimes \Pi_{\mathfrak{F}}, \mathfrak{A} \otimes \mathfrak{B})$.

2.2 Structure Manifolds and Zariski Tangent Spaces

Definition 2.5. *The structure manifold of an environment* $(\mathfrak{E}, \Pi, \mathfrak{A})$ *is the algebraic set* $\mathcal{S}(\mathfrak{E}) = \mathcal{S}(\mathfrak{E}, \Pi, \mathfrak{A}) \subseteq \mathfrak{E}$ *consisting of elements* $\alpha \in \mathfrak{E}$ *that satisfy the following algebraic equations,*

(i) $\Pi(\alpha) = 1_{\mathfrak{A}}$,

(ii) $\alpha \times \alpha = \alpha$,

(iii) $\Pi(\varphi \times \alpha) \cdot \alpha = \varphi \times \alpha, \quad \varphi \in \mathfrak{E}$,

(iv) $\alpha \cdot \Pi(\alpha \times \psi) = \alpha \times \psi, \quad \psi \in \mathfrak{E}$,

(v) $\Pi(\varphi \times \alpha)\, \Pi(\alpha \times \psi) = \Pi(\varphi \times \alpha \times \psi), \quad \varphi, \;\psi \in \mathfrak{E}$.

If $(\mathfrak{E}, \Pi, \mathfrak{A})$ *is involutive, we define* $\mathcal{S}_*(\mathfrak{E}) = \mathcal{S}_*(\mathfrak{E}, \Pi, \mathfrak{A}) \subseteq \mathcal{S}(\mathfrak{E})$ *as the algebraic subset of self–adjoint elements* $\alpha \in \mathcal{S}(\mathfrak{E})$*, i.e.,*

(vi) $\alpha^* = \alpha$.

We refer to each $\alpha \in \mathcal{S}(\mathfrak{E})$ *as a geometric* $\mathfrak{E}$*–structure on* $\mathfrak{A}$*.*

Assuming that $(\mathfrak{E}, \Pi_{\mathfrak{E}}, \mathfrak{A})$ and $(\mathfrak{F}, \Pi_{\mathfrak{F}}, \mathfrak{B})$ are two environments, we note that whenever $\alpha \in \mathcal{S}(\mathfrak{E})$, $\beta \in \mathcal{S}(\mathfrak{F})$, we get $\alpha \oplus \beta \in \mathcal{S}(\mathfrak{E} \oplus \mathfrak{F})$ and $\alpha \otimes \beta \in \mathcal{S}(\mathfrak{E} \otimes \mathfrak{F})$.

Our next goal is to define the Zariski tangent spaces to the structure manifolds $\mathcal{S}(\mathfrak{E})$ or $\mathcal{S}_*(\mathfrak{E})$ of an environment $(\mathfrak{E}, \Pi, \mathfrak{A})$. Following the same approach as in Martin [39], we rely on the algebra of real dual numbers, which is just the graded exterior algebra $\Lambda^{\#}(\mathbb{R}) = \Lambda^0(\mathbb{R}) \oplus \Lambda^1(\mathbb{R})$, with the usual algebraic structures, including an inner product, an involution, and a distinguished unit element $\delta \in \Lambda^1(\mathbb{R})$, the dual unit, satisfying the properties $\delta^* = \delta, \;\delta^2 = 0$. The set $\{1, \delta\}$, $1 \in \Lambda^0(\mathbb{R}) \equiv \mathbb{R}, \;\delta \in \Lambda^1(\mathbb{R}) \equiv \mathbb{R}\,\delta$, forms a linear basis for $\Lambda^{\#}(\mathbb{R}) \equiv \mathbb{R}[\delta]$. Any algebra $\mathfrak{U}$ has an extension $\mathfrak{U}[\delta] = \mathfrak{U} \otimes \mathbb{R}[\delta] = \mathfrak{U} + \mathfrak{U}\,\delta$. If $\mathfrak{U}$ is unital with unit $1_{\mathfrak{U}}$, then $\mathfrak{U}[\delta]$ is unital and $1_{\mathfrak{U}[\delta]} = 1_{\mathfrak{U}}$. If $\mathfrak{U}$ is involutive, the extension $\mathfrak{U}[\delta]$ is involutive, too. Consequently, each environment $(\mathfrak{E}, \Pi, \mathfrak{A})$ has an extension $(\mathfrak{E}[\delta], \Pi_{[\delta]}, \mathfrak{A}[\delta])$, with total algebra $\mathfrak{E}[\delta] = \{\, \alpha + \theta\,\delta : \alpha, \;\theta \in \mathfrak{E} \,\}$ and environment projection $\Pi_{[\delta]}(\alpha + \theta\,\delta) = \Pi(\alpha) + \Pi(\theta)\,\delta \in \mathfrak{A}[\delta]$.

Definition 2.6. *Let* $(\mathfrak{E}, \Pi, \mathfrak{A})$ *be an environment with structure manifold* $\mathcal{S}(\mathfrak{E})$*.*

(i) *The Zariski tangent space* $\mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})$ *to* $\mathcal{S}(\mathfrak{E})$ *at* $\alpha \in \mathcal{S}(\mathfrak{E})$ *consists of all elements* $\theta \in \mathfrak{E}$ *such that* $\alpha + \theta\,\delta \in \mathcal{S}(\mathfrak{E}[\delta])$*, the structure manifold associated with environment* $(\mathfrak{E}[\delta], \Pi_\delta, \mathfrak{A}[\delta])$*.*

(ii) *For an involutive environment, the Zariski tangent space* $\mathrm{T}_\alpha\,\mathcal{S}_*(\mathfrak{E})$ *to* $\mathcal{S}_*(\mathfrak{E})$ *at* $\alpha \in \mathcal{S}_*(\mathfrak{E})$ *is defined by assuming that* $\alpha + \theta\,\delta \in \mathcal{S}_*(\mathfrak{E}[\delta])$*.*

We refer to $\theta \in \mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})$*,* $\alpha \in \mathcal{S}(\mathfrak{E})$*, as a tangent* $\mathfrak{E}$*–structure on* $\mathfrak{A}$ *at* α*.*

Direct calculations lead to the next result derived from Definitions 2.5, 2.6.

Lemma 2.7. *If* $\alpha \in \mathcal{S}(\mathfrak{E})$ *and* $\theta \in \mathfrak{E}$*, then* $\theta \in \mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})$ *only if*

(i) $\Pi(\theta) = 0,$

(ii) $\theta \times \alpha + \alpha \times \theta = \theta,$

(iii) $\Pi(\varphi \times \theta) \cdot \alpha + \Pi(\varphi \times \alpha) \cdot \theta = \varphi \times \theta, \quad \varphi \in \mathfrak{E},$

(iv) $\theta \cdot \Pi(\alpha \times \psi) + \alpha \cdot \Pi(\theta \times \psi) = \theta \times \psi, \quad \psi \in \mathfrak{E},$

(v) $\Pi(\varphi \times \theta)\,\Pi(\alpha \times \psi) + \Pi(\varphi \times \alpha)\,\Pi(\theta \times \psi) = \Pi(\varphi \times \theta \times \psi), \quad \varphi, \psi \in \mathfrak{E}.$

If $(\mathfrak{E}, \Pi, \mathfrak{A})$ *is involutive and* $\alpha \in \mathcal{S}_*(\mathfrak{E})$*, then* $\theta \in \mathrm{T}_\alpha\,\mathcal{S}_*(\mathfrak{E})$ *provided, in addition,*

(vi) $\theta^* = \theta.$

Proof. Property (i) follows from requirement (i) in Definition 2.5. Assuming that $\Pi_{[\delta]}(\alpha + \theta\,\delta) = 1_{\mathfrak{A}[\delta]} = 1_{\mathfrak{A}}$ we get

$$1_{\mathfrak{A}} = \Pi(\alpha) + \Pi(\theta)\,\delta = 1_{\mathfrak{A}} + \Pi(\theta)\,\delta, \text{ i.e., } \Pi(\theta) = 0.$$

Property (ii) is a consequence of $(\alpha + \theta\,\delta) \times (\alpha + \theta\,\delta) = \alpha + \theta\,\delta$, which is requirement (ii) in Definition 2.5. The other properties, (iii) − (vi), have similar straightforward proofs. □

We note that the list of requirements in Definition 2.5 is redundant. For instance, (i) and (ii) imply $\Pi(\alpha \times \alpha) = 1_{\mathfrak{A}}$, from which both requirements are recovered based on either (iii) or (iv). At the same time, requirement (v) is a consequence of (iii) and (iv). Moreover, if environment $(\mathfrak{E}, \Pi, \mathfrak{A})$ is involutive and $\alpha \in \mathcal{S}(\mathfrak{E})$, then $\alpha^* \in \mathcal{S}(\mathfrak{E})$, and consequently requirements (iii) and (iv) imply each other. The previous remarks indicate that the list of properties in Lemma 2.7 is redundant, too. Eventually, we will single out two minimal lists and give alternate characterizations of structure manifolds and Zariski tangent spaces.

With regard to functorial properties, if $(\Lambda, \Lambda_0) : (\mathfrak{E}, \Pi_{\mathfrak{E}}, \mathfrak{A}) \to (\mathfrak{F}, \Pi_{\mathfrak{F}}, \mathfrak{B})$ is an environment homomorphism and $\Lambda : \mathfrak{E} \to \mathfrak{F}$ is onto, Definition 2.5 implies $\Lambda(\mathcal{S}(\mathfrak{E})) \subseteq \mathcal{S}(\mathfrak{F})$, $\Lambda(\mathcal{S}_*(\mathfrak{E})) \subseteq \mathcal{S}_*(\mathfrak{F})$, and if $\alpha \in \mathcal{S}(\mathfrak{E})$, $\beta = \Lambda(\alpha) \in \mathcal{S}(\mathfrak{F})$, from Lemma 2.7 we get $\Lambda(\mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})) \subseteq \mathrm{T}_\beta\,\mathcal{S}(\mathfrak{F})$, $\Lambda(\mathrm{T}_\alpha\,\mathcal{S}_*(\mathfrak{E})) \subseteq \mathrm{T}_\beta\,\mathcal{S}_*(\mathfrak{F})$.

2.3 Derivations on Algebra Environments

We start by recalling that a derivation on an algebra $\mathfrak{U}$ is a linear mapping $\mathcal{D} : \mathfrak{U} \to \mathfrak{U}$ with the property $\mathcal{D}(u\,v) = \mathcal{D}(u)\,v + u\,\mathcal{D}(v)$, $u, v \in \mathfrak{U}$. If $\mathfrak{U}$ is involutive and $\mathcal{D}(u^*) = \mathcal{D}(u)^*$, $u \in \mathfrak{U}$, we refer to $\mathcal{D}$ as an involution preserving derivation. The space $\mathrm{Der}(\mathfrak{U})$ of derivations on $\mathfrak{U}$ is a Lie algebra with the Lie product $[\,\cdot\,,\cdot\,]$ given by the commutator of two derivations,

$$[\,\mathcal{D}, \mathcal{D}'\,] = \mathcal{D}\,\mathcal{D}' - \mathcal{D}'\,\mathcal{D}, \quad \mathcal{D},\, \mathcal{D}' \in \mathrm{Der}(\mathfrak{U}).$$

The subspace $\mathrm{Der}_*(\mathfrak{U})$ of involution preserving derivations is a Lie subalgebra.

Definition 2.8. *A derivation on an environment* $(\mathfrak{E}, \Pi, \mathfrak{A})$ *is a pair* $(\mathcal{D}, \mathcal{D}_0)$*, where* $\mathcal{D} \in \mathrm{Der}(\mathfrak{E})$ *and* $\mathcal{D}_0 \in \mathrm{Der}(\mathfrak{A})$*, such that*

$$\mathcal{D}(a \cdot \varphi) = \mathcal{D}_0(a) \cdot \varphi + a \cdot \mathcal{D}(\varphi), \quad \mathcal{D}(\varphi \cdot a) = \mathcal{D}(\varphi) \cdot a + \varphi \cdot \mathcal{D}_0(a), \ a \in \mathfrak{A}, \ \varphi \in \mathfrak{E},$$

$$\Pi(\mathcal{D}(\varphi)) = \mathcal{D}_0(\Pi(\varphi)), \ \varphi \in \mathfrak{E}.$$

For involutive environments we may require $\mathcal{D}$*,* $\mathcal{D}_0$ *to be involution preserving.*

The space $\mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ of derivations on $(\mathfrak{E}, \Pi, \mathfrak{A})$ is a Lie algebra, and the subspace $\mathrm{Der}_*(\mathfrak{E}, \Pi, \mathfrak{A})$ of involution preserving derivations is a Lie subalgebra.

Lemma 2.9. *Let* $(\mathcal{D}, \mathcal{D}_0) \in \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ *be a derivation on* $(\mathfrak{E}, \Pi, \mathfrak{A})$*.*

(i) *If* $\alpha \in \mathcal{S}(\mathfrak{E})$*, then* $\theta = \mathcal{D}(\alpha) \in \mathrm{T}_\alpha \mathcal{S}(\mathfrak{E})$*.*

(ii) *If* $(\mathfrak{E}, \Pi, \mathfrak{A})$ *is involutive,* $(\mathcal{D}, \mathcal{D}_0) \in \mathrm{Der}_*(\mathfrak{E}, \Pi, \mathfrak{A})$*, and* $\alpha \in \mathcal{S}_*(\mathfrak{E})$*, then* $\theta = \mathcal{D}(\alpha) \in \mathrm{T}_\alpha \mathcal{S}_*(\mathfrak{E})$*.*

Proof. To prove both statements we have to show that $\theta = \mathcal{D}(\alpha)$ satisfies the properties in Lemma 2.7. To this end, we apply derivation $(\mathcal{D}, \mathcal{D}_0)$ to each equation in Definition 2.5. Property (i) in Lemma 2.7 for instance follows from requirement (i) in Definition 2.5 by observing that

$$\Pi(\theta) = \Pi(\mathcal{D}(\alpha)) = \mathcal{D}_0(\Pi(\alpha)) = \mathcal{D}_0(1_{\mathfrak{A}}) = 0.$$

The proof of property (ii) in Lemma 2.7 is a consequence of requirement (ii) in Definition 2.5, and goes as follows,

$$\theta \times \alpha + \alpha \times \theta = \mathcal{D}(\alpha) \times \alpha + \alpha \times \mathcal{D}(\alpha) = \mathcal{D}(\alpha \times \alpha) = \mathcal{D}(\alpha) = \theta.$$

The other properies, (iii) – (vi), are derived in a similar way. □

Prompted by Lemma 2.9 we proceed with a few definitions of some new tools. Let $(\mathfrak{E}, \Pi, \mathfrak{A})$ be an environment and assign to each $\alpha \in \mathcal{S}(\mathfrak{E})$ the linear operator

$$\Delta_\alpha : \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}) \to \mathfrak{E}, \ \Delta_\alpha(\mathcal{D}, \mathcal{D}_0) = \mathcal{D}(\alpha), \ (\mathcal{D}, \mathcal{D}_0) \in \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}). \tag{2.1}$$

If $(\mathfrak{E}, \Pi, \mathfrak{A})$ is involutive and $\alpha \in \mathcal{S}_*(\mathfrak{E})$, we let $\Delta_{\alpha,*} : \mathrm{Der}_*(\mathfrak{E}, \Pi, \mathfrak{A}) \to \mathfrak{E}_{\mathrm{h}}$ be the restriction of Δ_α to $\mathrm{Der}_*(\mathfrak{E}, \Pi, \mathfrak{A})$ and note that its values are elements of $\mathfrak{E}_{\mathrm{h}} = \{ \varphi \in \mathfrak{E} : \varphi^* = \varphi \}$. By parts (i) and (ii) of Lemma 2.9 we obviously get

$$\mathrm{Ran}(\Delta_\alpha) = \{ \theta \in \mathfrak{E} : \theta = \mathcal{D}(\alpha), \ (\mathcal{D}, \mathcal{D}_0) \in \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}) \} \subseteq \mathrm{T}_\alpha \mathcal{S}(\mathfrak{E}),$$

$$\mathrm{Ran}(\Delta_{\alpha,*}) = \{ \theta \in \mathfrak{E}_{\mathrm{h}} : \theta = \mathcal{D}(\alpha), \ (\mathcal{D}, \mathcal{D}_0) \in \mathrm{Der}_*(\mathfrak{E}, \Pi, \mathfrak{A}) \} \subseteq \mathrm{T}_\alpha \mathcal{S}_*(\mathfrak{E}).$$

Actually, further investigations will show that $\mathrm{Ran}(\Delta_\alpha)$ and $\mathrm{Ran}(\Delta_{\alpha,*})$ yield the spaces $\mathrm{T}_\alpha \mathcal{S}(\mathfrak{E})$ and $\mathrm{T}_\alpha \mathcal{S}_*(\mathfrak{E})$ in their entirety.

We next introduce the Lie subalgebra $\mathrm{Der}_{\mathrm{inn}}(\mathfrak{E}, \Pi, \mathfrak{A}) \subseteq \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ of inner derivations on $(\mathfrak{E}, \Pi, \mathfrak{A})$, $\mathrm{Der}_{\mathrm{inn}}(\mathfrak{E}, \Pi, \mathfrak{A}) = \{ (\mathcal{D}_x, \mathcal{D}_{0,x}) : x \in \mathfrak{A} \}$, where

$$\mathcal{D}_{0,x} \in \mathrm{Der}(\mathfrak{A}), \quad \mathcal{D}_{0,x}(a) = [\, x \,, a \,] = x\, a - a\, x, \quad x \in \mathfrak{A}, \ a \in \mathfrak{A},$$

$$\mathcal{D}_x \in \mathrm{Der}(\mathfrak{E}), \quad \mathcal{D}_x(\varphi) = [\, x \,, \varphi \,] = x \,\cdot \varphi - \varphi \,\cdot\, x, \quad x \in \mathfrak{A}, \ \varphi \in \mathfrak{E}.$$

For a latter use, we define the linear mappings

$$\mathfrak{D} : \mathfrak{A} \to \mathrm{Der}(\mathfrak{E}),\ \mathfrak{D}(x) = \mathcal{D}_x, \quad \mathfrak{D}_0 : \mathfrak{A} \to \mathrm{Der}(\mathfrak{A}),\ \mathfrak{D}_0(x) = \mathcal{D}_{0,x}, \quad x \in \mathfrak{A},$$

and the operator

$$(\mathfrak{D}, \mathfrak{D}_0) : \mathfrak{A} \to \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}), \quad (\mathfrak{D}, \mathfrak{D}_0)(x) = (\mathfrak{D}(x), \mathfrak{D}_0(x)), \quad x \in \mathfrak{A}. \tag{2.2}$$

We note that $(\mathcal{D}_x, \mathcal{D}_{0,x}) \in \mathrm{Der}_*(\mathfrak{E}, \Pi, \mathfrak{A})$ only if $x^* = -x$.
The following definition introduces another quite useful concept.

Definition 2.10. *Let* $(\mathfrak{E}, \Pi, \mathfrak{A})$ *be an algebra environment. The symbol operator associated with a structure* $\alpha \in \mathcal{S}(\mathfrak{E})$ *is defined by*

$$\Sigma_\alpha : \mathfrak{E} \to \mathfrak{A}, \quad \Sigma_\alpha(\varphi) = 2^{-1}\, \Pi(\varphi \times \alpha - \alpha \times \varphi), \quad \varphi \in \mathfrak{E}. \tag{2.3}$$

If $(\mathfrak{E}, \Pi, \mathfrak{A})$ *is involutive and* $\alpha \in \mathcal{S}_*(\mathfrak{E})$*, the restricted symbol operator is*

$$\Sigma_{\alpha,*} : \mathfrak{E}_{\mathrm{h}} \to \mathfrak{A}_{\mathrm{sh}}, \quad \Sigma_{\alpha,*} = \Sigma_\alpha | \mathfrak{E}_{\mathrm{h}}, \quad \mathfrak{A}_{\mathrm{sh}} = \{\, x \in \mathfrak{A} : x^* = -x \,\}.$$

Related to any environment $(\mathfrak{E}, \Pi_{\mathfrak{E}}, \mathfrak{A})$ and each structure $\alpha \in \mathcal{S}(\mathfrak{E})$ we define a subalgebra $\mathfrak{A}_\alpha \subseteq \mathfrak{A}$ of base algebra $\mathfrak{A}$ by

$$\mathfrak{A}_\alpha = \{\, a \in \mathfrak{A} : a \cdot \alpha = \alpha \cdot a \,\}, \tag{2.4}$$

and the linear operator $\pi_\alpha : \mathfrak{A} \to \mathfrak{A}$ given by

$$\pi_\alpha(x) = \Pi(\alpha \cdot x \times \alpha) = \Pi(\alpha \times x \cdot \alpha), \quad x \in \mathfrak{A}. \tag{2.5}$$

If $(\mathfrak{E}, \Pi, \mathfrak{A})$ is an involutive environment and $\alpha \in \mathcal{S}_*(\mathfrak{E})$, subalgebra $\mathfrak{A}_\alpha$ is involutive and $\pi_\alpha(x^*) = \pi_\alpha(x)^*$, $x \in \mathfrak{A}$.

Lemma 2.11. *The operator* $\pi_\alpha : \mathfrak{A} \to \mathfrak{A}$ *is a projection onto subalgebra* $\mathfrak{A}_\alpha$*, i.e.,* $\pi_\alpha^2 = \pi_\alpha,\ \mathrm{Ran}(\pi_\alpha) = \mathfrak{A}_\alpha$*, with the additional properties*

$$\pi_\alpha(1_{\mathfrak{A}}) = 1_{\mathfrak{A}},\ \pi_\alpha(a\, x) = a\, \pi_\alpha(x),\ \pi_\alpha(x\, a) = \pi_\alpha(x)\, a,\ a \in \mathfrak{A}_\alpha,\ x \in \mathfrak{A}. \tag{2.6}$$

Essentially, Lemma 2.11 points out that each $\pi_\alpha : \mathfrak{A} \to \mathfrak{A},\ \alpha \in \mathcal{S}(\mathfrak{E})$, is a non–commutative conditional expectation from algebra $\mathfrak{A}$ onto subalgebra $\mathfrak{A}_\alpha$.

Proof. We rely on defining properties of geometric structures. We first observe that requirements (iii) and (iv) in Definition 2.5 imply

$$\pi_\alpha(x) \cdot \alpha = \Pi(\alpha \cdot x \times \alpha) \cdot \alpha = \alpha \cdot x \times \alpha = \alpha \times x \cdot \alpha = \alpha \cdot \Pi(\alpha \times x \cdot \alpha) = \alpha \cdot \pi_\alpha(x),$$

hence $\mathrm{Ran}(\pi_\alpha) \subseteq \mathfrak{A}_\alpha$, and then by assuming that $a \in \mathfrak{A}_\alpha$ we get

$$\pi_\alpha(a) = \Pi(\alpha \cdot a \times \alpha) = \Pi(a \cdot \alpha \times \alpha) = \Pi(a \cdot \alpha) = a\, \Pi(\alpha) = a\, 1_{\mathfrak{A}} = a.$$

The proof of the the first equation in statement (2.6) reduces to

$$\pi_\alpha(1_{\mathfrak{A}}) = \Pi(\alpha \cdot 1_{\mathfrak{A}} \times \alpha) = \Pi(\alpha \times \alpha) = \Pi(\alpha) = 1_{\mathfrak{A}},$$

and for the two other equations, if $a \in \mathfrak{A}_\alpha$ and $x \in \mathfrak{A}$, we have

$$\pi_\alpha(a\,x) = \Pi(\alpha \cdot a\,x \times \alpha) = \Pi(a \cdot \alpha \cdot x \times \alpha) = a\,\Pi(\alpha \cdot x \times \alpha) = a\,\pi_\alpha(x),$$

$$\pi_\alpha(x\,a) = \Pi(\alpha \times x\,a \cdot \alpha) = \Pi(\alpha \times x \cdot \alpha \cdot a) = \Pi(\alpha \times x \cdot \alpha)\,a = \pi_\alpha(x)\,a.$$

The proof is complete. ☐

We note that each subalgebra $\mathfrak{A}_\alpha$, $\alpha \in \mathcal{S}(\mathfrak{E})$, has a direct complement, the subspace $\mathfrak{A}_\alpha^\perp = \{x \in \mathfrak{A} : \pi_\alpha(x) = 0\}$, range of the complement projection $\pi_\alpha^\perp : \mathfrak{A} \to \mathfrak{A}$, $\pi_\alpha^\perp = \mathrm{Id}_{\mathfrak{A}} - \pi_\alpha$.

The next technical result records consequences of the previous definitions.

Lemma 2.12. *Let* $(\mathfrak{E}, \Pi, \mathfrak{A})$ *be an environment and* $\alpha \in \mathcal{S}(\mathfrak{E})$. *The three linear operators,* $\Delta_\alpha : \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}) \to \mathfrak{E}$, $(\mathfrak{D}, \mathfrak{D}_0) : \mathfrak{A} \to \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$, $\Sigma_\alpha : \mathfrak{E} \to \mathfrak{A}$, *and the projections* π_α, $\pi_\alpha^\perp : \mathfrak{A} \to \mathfrak{A}$, *have the following properties.*

(i) $\pi_\alpha \circ \Sigma_\alpha : \mathfrak{E} \to \mathfrak{A}$ *and* $\Sigma_\alpha \circ \Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0) : \mathfrak{A} \to \mathfrak{A}$ *satisfy*

$$\pi_\alpha \circ \Sigma_\alpha = 0, \quad \Sigma_\alpha \circ \Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0) = \pi_\alpha^\perp.$$

Consequently, $\Sigma_\alpha = \pi_\alpha^\perp \circ \Sigma_\alpha$ *and* $\mathrm{Ran}\,(\Sigma_\alpha) = \mathfrak{A}_\alpha^\perp$.

(ii) *The kernel and range of* $\Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0) : \mathfrak{A} \to \mathfrak{E}$ *are given by*

$$\mathrm{Ker}\,(\Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0)) = \mathfrak{A}_\alpha, \quad \mathrm{Ran}\,(\Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0)) = \mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E}).$$

If $(\mathfrak{E}, \Pi, \mathfrak{A})$ *is an involutive environment and* $\alpha \in \mathcal{S}_*(\mathfrak{E})$, *parts* (i) *and* (ii) *have the next counterparts.*

(iii) $\pi_\alpha \circ \Sigma_{\alpha,*} : \mathfrak{E}_{\mathrm{h}} \to \mathfrak{A}_{\mathrm{sh}}$ *and* $\Sigma_{\alpha,*} \circ \Delta_{\alpha,*} \circ (\mathfrak{D}, \mathfrak{D}_0)|\,\mathfrak{A}_{\mathrm{sh}} : \mathfrak{A}_{\mathrm{sh}} \to \mathfrak{A}_{\mathrm{sh}}$ *satisfy*

$$\pi_\alpha \circ \Sigma_{\alpha,*} = 0, \quad \Sigma_\alpha \circ \Delta_{\alpha,*} \circ (\mathfrak{D}, \mathfrak{D}_0)|\,\mathfrak{A}_{\mathrm{sh}} = \pi_\alpha^\perp|\,\mathfrak{A}_{\mathrm{sh}}.$$

Therefore, $\mathrm{Ran}\,(\Sigma_{\alpha,*}) = \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}}$.

(iv) *The kernel and range of* $\Delta_{\alpha,*} \circ (\mathfrak{D}, \mathfrak{D}_0)|\,\mathfrak{A}_{\mathrm{sh}} : \mathfrak{A}_{\mathrm{sh}} \to \mathfrak{E}_{\mathrm{h}}$ *are given by*

$$\mathrm{Ker}\,(\Delta_{\alpha,*} \circ (\mathfrak{D}, \mathfrak{D}_0)|\,\mathfrak{A}_{\mathrm{sh}}) = \mathfrak{A}_\alpha \cap \mathfrak{A}_{\mathrm{sh}}, \quad \mathrm{Ran}\,(\Delta_{\alpha,*} \circ (\mathfrak{D}, \mathfrak{D}_0)|\,\mathfrak{A}_{\mathrm{sh}}) = \mathrm{T}_\alpha\,\mathcal{S}_*(\mathfrak{E}).$$

Proof. Suppose $\varphi \in \mathfrak{E}$ and let $x = \Sigma_\alpha(\varphi) \in \mathfrak{A}$. Using equation (2.3) and requirements (iii) and (iv) in Definition 2.5 we get

$$\begin{aligned} \alpha \times x \cdot \alpha &= 2^{-1}\,\alpha \times \Pi(\varphi \times \alpha - \alpha \times \varphi) \cdot \alpha \\ &= 2^{-1}\,[\,\alpha \times \Pi(\varphi \times \alpha) \cdot \alpha - \alpha \cdot \Pi(\alpha \times \varphi) \times \alpha\,] \\ &= 2^{-1}\,(\alpha \times \varphi \times \alpha - \alpha \times \varphi \times \alpha) = 0, \end{aligned}$$

hence $\pi_\alpha(x) = \Pi(\alpha \times x \cdot \alpha) = 0$. Next, given $x \in \mathfrak{A}$, let $\varphi = \Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0)(x)$, i.e., $\varphi = x \cdot \alpha - \alpha \cdot x \in \mathfrak{E}$, and observe that

$$\varphi \times \alpha - \alpha \times \varphi = (x \cdot \alpha - \alpha \cdot x) \times \alpha - \alpha \times (x \cdot \alpha - \alpha \cdot x) = x \cdot \alpha - 2\,\alpha \times x \cdot \alpha + \alpha \cdot x.$$

Since $\Pi(\alpha) = 1$, we have

$$\Sigma_\alpha(\varphi) = 2^{-1}\,\Pi(\varphi \times \alpha - \alpha \times \varphi) = x - \Pi(\alpha \times x \cdot \alpha) = x - \pi_\alpha(x) = \pi_\alpha^\perp(x).$$

We just proved statement (i). Parts of statement (ii), $\operatorname{Ker}(\Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0)) = \mathfrak{A}_\alpha$ and $\operatorname{Ran}(\Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0)) \subseteq \mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})$, are consequences of the definition of $\mathfrak{A}_\alpha$ and Lemma 2.9. Assume now that $\theta \in \mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})$, and let $x = \Sigma_\alpha(\theta) \in \mathfrak{A}_\alpha^\perp$. By properties (i) and (ii) in Lemma 2.7 we know that

$$\Pi(\theta) = 0, \quad \theta \times \alpha + \alpha \times \theta = \theta.$$

Consequently, $\theta \times \alpha - \alpha \times \theta = 2\,\theta \times \alpha - \theta = \theta - 2\,\alpha \times \theta$, and (2.3) implies

$$x = \Sigma_\alpha(\theta) = \Pi(\theta \times \alpha) = -\,\Pi(\alpha \times \theta). \tag{2.7}$$

Using once more requirements (iii) and (iv) in Definition 2.5, from (2.7) we get

$$\Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0)(x) = x \cdot \alpha - \alpha \cdot x = \Pi(\theta \times \alpha) \cdot \alpha + \alpha \cdot \Pi(\alpha \times \theta) = \theta \times \alpha + \alpha \times \theta = \theta.$$

The proof of statement (ii) is concluded.

Statements (iii) and (iv) follow from (i) and (ii). Since $\alpha^* = \alpha$, we note that

$$\Sigma_\alpha(\varphi)^* = -\,\Sigma_\alpha(\varphi^*), \quad (x \cdot \alpha - \alpha \cdot x)^* = -\,(x^* \cdot \alpha - \alpha \cdot x^*),$$

for any $\varphi \in \mathfrak{E}$ and $x \in \mathfrak{A}_\alpha$. Therefore, if $\varphi \in \mathfrak{E}_{\mathrm{h}}$, then $x = \Sigma_\alpha(\varphi) \in \mathfrak{A}_{\mathrm{sh}}$, and whenever $x \in \mathfrak{A}_{\mathrm{sh}}$, then $\varphi = x \cdot \alpha - \alpha \cdot x \in \mathfrak{E}_{\mathrm{h}}$, with $\Sigma_\alpha(\varphi) = x$. □

A quick inspection of the proof of Lemma 2.12 shows that the Zariski tangent spaces to structure manifolds are completely determined by only two, or three, properties in Lemma 2.7.

Corollary 2.13. *Suppose $\alpha \in \mathcal{S}(\mathfrak{E})$, or $\alpha \in \mathcal{S}_*(\mathfrak{E})$, and $\theta \in \mathfrak{E}$.*

(i) $\theta \in \mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})$ *only if* $\Pi(\theta) = 0$ *and* $\theta \times \alpha + \alpha \times \theta = \theta$.

(ii) $\theta \in \mathrm{T}_\alpha\,\mathcal{S}_*(\mathfrak{E})$ *if, in addition,* $\theta^* = \theta$. □

We are now in a position to give two refined descriptions of tangent spaces.

Theorem A – Zariski Tangent Spaces to $\mathcal{S}(\mathfrak{E})$ and $\mathcal{S}_*(\mathfrak{E})$

Let $(\mathfrak{E}, \Pi, \mathfrak{A})$ be an algebra environment and assume that $\alpha \in \mathcal{S}(\mathfrak{E})$.

(i) *The linear mapping* $\mathrm{T}_\alpha : \mathfrak{A}_\alpha^\perp \to \mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})$ *defined by*

$$\mathrm{T}_\alpha(x) = \Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0)(x) = x \cdot \alpha - \alpha \cdot x, \quad x \in \mathfrak{A}_\alpha^\perp, \tag{2.8}$$

is a vector space isomorphism.

(ii) *The operator* $\mathrm{P}_\alpha : \mathfrak{E} \to \mathfrak{E}$ *given by* $\mathrm{P}_\alpha = \Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0) \circ \Sigma_\alpha$, *i.e.,*

$$\mathrm{P}_\alpha(\varphi) = \Sigma_\alpha(\varphi) \cdot \alpha - \alpha \cdot \Sigma_\alpha(\varphi), \quad \varphi \in \mathfrak{E}, \tag{2.9}$$

is a projection of $\mathfrak{E}$ onto the Zariski tangent space $\mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})$.

Suppose next that $(\mathfrak{E}, \Pi, \mathfrak{A})$ is an involutive algebra environment and $\alpha \in \mathcal{S}_*(\mathfrak{E})$.

(iii) *The linear mapping* $\mathrm{T}_{\alpha,*} : \mathfrak{A}_\alpha^\perp \cap \mathfrak{A}_{\mathrm{sh}} \to \mathrm{T}_\alpha\, \mathcal{S}_*(\mathfrak{E})$ *defined as the restriction of* T_α *is a vector space isomorphism.*

(iv) *The operator* $\mathrm{P}_{\alpha,*} : \mathfrak{E}_{\mathrm{h}} \to \mathfrak{E}_{\mathrm{h}}$ *defined as the restriction of* P_α *is a projection of* $\mathfrak{E}_{\mathrm{h}}$ *onto the Zariski tangent space* $\mathrm{T}_\alpha\, \mathcal{S}_*(\mathfrak{E})$.

Proof. Statement (i) is a consequence of part (ii) in Lemma 2.12. Statement (ii) is yet another consequence. The equations in part (i) of Lemma 2.12 imply

$$\begin{aligned}\mathrm{P}_\alpha^2 &= \Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0) \circ [\, \Sigma_\alpha \circ \Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0)\,] \circ \Sigma_\alpha \\ &= \Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0) \circ \pi_\alpha^\perp \circ \Sigma_\alpha = \Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0) \circ \Sigma_\alpha = \mathrm{P}_\alpha,\end{aligned}$$

hence P_α is a projection, and obviously $\mathrm{Ran}(\mathrm{P}_\alpha) \subseteq \mathrm{T}_\alpha\, \mathcal{S}(\mathfrak{E})$ by Lemma 2.9. On the other hand, if $\theta \in \mathrm{T}_\alpha\, \mathcal{S}(\mathfrak{E})$ and $x = \Sigma_\alpha(\theta)$, Lemma 2.12 implies $\mathrm{P}_\alpha(\theta) = \theta$. The last two observations clearly show that $\mathrm{Ran}\, \mathrm{P}_\alpha = \mathrm{T}_\alpha\, \mathcal{S}(\mathfrak{E})$.

Statements (iii), (iv) follow from (i), (ii). The proof is complete. □

The remaining part of this section analyzes special classes of derivations.

Definition 2.14. *A derivation* $(\mathcal{D}, \mathcal{D}_0) \in \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ *is called compatible with a structure* $\alpha \in \mathcal{S}(\mathfrak{E})$ *provided* $\Delta_\alpha(\mathcal{D}, \mathcal{D}_0) = 0$, *i.e.*, $\mathcal{D}(\alpha) = 0$. *The Lie subalgebra of derivations compatible with* $\alpha \in \mathcal{S}(\mathfrak{E})$ *is denoted by* $\mathrm{Der}_{\alpha,0}(\mathfrak{E}, \Pi, \mathfrak{A})$.

Definition 2.15. *The conjugation operator* $\Gamma_\alpha : \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}) \to \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ *associated with* $\alpha \in \mathcal{S}(\mathfrak{E})$ *is defined as*

$$\Gamma_\alpha = \mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})} - 2\, (\mathfrak{D}, \mathfrak{D}_0) \circ \Sigma_\alpha \circ \Delta_\alpha. \tag{2.10}$$

Theorem B – Conjugation Operator and Compatible Derivations

The operator $\Gamma_\alpha : \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A}) \to \mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$ *has the following properties.*

(i) Γ_α *is an involution on* $\mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$, *i.e.*, $\Gamma_\alpha^2 = \mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})}$.

(ii) $\Delta_\alpha \circ \Gamma_\alpha = -\,\Delta_\alpha$.

(iii) $(\mathcal{D}, \mathcal{D}_0) \in \mathrm{Der}_{\alpha,0}(\mathfrak{E}, \Pi, \mathfrak{A})$ *only if* $\Gamma_\alpha(\mathcal{D}, \mathcal{D}_0) = (\mathcal{D}, \mathcal{D}_0)$.

Proof. Using the restriction of Σ_α to $\mathrm{T}_\alpha\, \mathcal{S}(\mathfrak{E})$, from Lemma 2.12 we get that

$$\Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0) \circ \Sigma_\alpha \,|\, \mathrm{T}_\alpha\, \mathcal{S}(\mathfrak{E}) = \mathrm{Id}_{\mathrm{T}_\alpha\, \mathcal{S}(\mathfrak{E})}. \tag{2.11}$$

Statements (i) and (ii) are both direct consequences. Specifically, we note that

$$\begin{aligned}\Gamma_\alpha^2 &= \mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})} - 4\, (\mathfrak{D}, \mathfrak{D}_0) \circ \Sigma_\alpha \circ \Delta_\alpha + 4\, [\, (\mathfrak{D}, \mathfrak{D}_0) \circ \Sigma_\alpha \circ \Delta_\alpha\,]^2 \\ &= \mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})} - 4\, (\mathfrak{D}, \mathfrak{D}_0) \circ \Sigma_\alpha \circ \Delta_\alpha + 4\, (\mathfrak{D}, \mathfrak{D}_0) \circ \Sigma_\alpha \circ \mathrm{Id}_{\mathrm{T}_\alpha\, \mathcal{S}(\mathfrak{E})} \circ \Delta_\alpha,\end{aligned}$$

hence $\Gamma_\alpha^2 = \mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})}$, and then observe that

$$\Delta_\alpha \circ \Gamma_\alpha = \Delta_\alpha - 2\, \Delta_\alpha \circ (\mathfrak{D}, \mathfrak{D}_0) \circ \Sigma_\alpha \circ \Delta_\alpha = \Delta_\alpha - 2\, \mathrm{Id}_{\mathrm{T}_\alpha\, \mathcal{S}(\mathfrak{E})} \circ \Delta_\alpha = -\,\Delta_\alpha.$$

Statement (iii), which identifies $\mathrm{Der}_{\alpha,0}(\mathfrak{E}, \Pi, \mathfrak{A})$ with the space of self–conjugate derivations in $\mathrm{Der}(\mathfrak{E}, \Pi, \mathfrak{A})$, i.e., the eigenspace of Γ_α corresponding to the eigenvalue $\lambda^+ = 1$, follows from Definition 2.14 and the previous results. If $(\mathcal{D}, \mathcal{D}_0) \in \mathrm{Der}_{\alpha,0}(\mathfrak{E}, \Pi, \mathfrak{A})$, then $\Delta_\alpha\, (\mathcal{D}, \mathcal{D}_0) = 0$ and from equation (2.10) we get $\Gamma_\alpha(\mathcal{D}, \mathcal{D}_0) = (\mathcal{D}, \mathcal{D}_0)$. Conversely, if $\Gamma_\alpha(\mathcal{D}, \mathcal{D}_0) = (\mathcal{D}, \mathcal{D}_0)$, statement (ii) shows that $\Delta_\alpha\, (\mathcal{D}, \mathcal{D}_0) = -\,\Delta_\alpha\, (\mathcal{D}, \mathcal{D}_0)$, hence $\Delta_\alpha\, (\mathcal{D}, \mathcal{D}_0) = 0$. The proof of Theorem B is complete. □

Corollary 2.16. *The operators* Γ_α^+, $\Gamma_\alpha^- : \mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A}) \to \mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})$,

$$\Gamma_\alpha^+ = (\,\mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})} + \Gamma_\alpha\,)/2, \quad \Gamma_\alpha^- = (\,\mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})} - \Gamma_\alpha\,)/2),$$

are complementary projections on the space $\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})$*, i.e.,*

$$(\Gamma_\alpha^+)^2 = \Gamma_\alpha^+,\ (\Gamma_\alpha^-)^2 = \Gamma_\alpha^-,\ \Gamma_\alpha^+\,\Gamma_\alpha^- = \Gamma_\alpha^-\,\Gamma_\alpha^+ = 0,\ \Gamma_\alpha^+ + \Gamma_\alpha^- = \mathrm{Id}_{\mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A})},$$

such that $\mathrm{Der}_{\alpha,0}(\mathfrak{E},\Pi,\mathfrak{A}) = \mathrm{Ran}\,(\Gamma_\alpha^+) = \mathrm{Ker}\,(\Gamma_\alpha^-)$. □

Corollary 2.17. *Suppose* $\theta \in \mathrm{T}_\alpha\,\mathcal{S}(\mathfrak{E})$ *and let* $\mathrm{Der}_{\alpha,\theta}(\mathfrak{E},\Pi,\mathfrak{A})$ *be defined as*

$$\mathrm{Der}_{\alpha,\theta}\,(\mathfrak{E},\Pi,\mathfrak{A}) = \{(\mathcal{D},\mathcal{D}_0) \in \mathrm{Der}(\mathfrak{E},\Pi,\mathfrak{A}) : \Delta_\alpha\,(\mathcal{D},\mathcal{D}_0) = \theta\}.$$

Then $\mathrm{Der}_{\alpha,\theta}(\mathfrak{E},\Pi,\mathfrak{A}) = (\mathfrak{D},\mathfrak{D}_0)\,\circ\,\Sigma_\alpha(\theta) + \mathrm{Der}_{\alpha,0}\,(\mathfrak{E},\Pi,\mathfrak{A})$. □

Theorem B and its Corollaries have natural counterparts for involutive algebra environments and spaces of involution preserving derivations derived by making some adjustments under appropriate assumptions.

3 Geometric Algebras and Hom Environments

This section introduces the class of geometric algebras and the concept of Hom environments $\mathfrak{H}[\,\mathfrak{S},\mathfrak{A}\,]$ associated with a geometric algebra $\mathfrak{S}$ and a unital involutive algebra $\mathfrak{A}$. The main result identifies the structure manifolds $\mathcal{S}(\mathfrak{H}[\,\mathfrak{S},\mathfrak{A}\,])$ and $\mathcal{S}_*(\mathfrak{H}[\,\mathfrak{S},\mathfrak{A}\,])$ with the spaces $\mathrm{Hom}_{\mathrm{alg}}(\mathfrak{S},\mathfrak{A})$ and $\mathrm{Hom}_{\mathrm{alg},*}(\mathfrak{S},\mathfrak{A})$ of algebra homomorphisms, or involutive homomorphisms, from $\mathfrak{S}$ to $\mathfrak{A}$. Consequently, we will be able to analyze such spaces of algebra homomorphisms by implementing techniques and results from Section 2.

3.1 Geometric Algebras — Definitions and Case Studies

We begin by assuming that $\mathfrak{S}$ is a finite dimensional unital involutive algebra with a faithful trace, i.e., a linear mapping $\tau : \mathfrak{S} \to \mathbb{R}$ such that

(i) $\tau(u\,v) = \tau(v\,u), \quad u, v \in \mathfrak{S}$,

(ii) $\tau(u^*) = \tau(u), \quad u \in \mathfrak{S}$,

(iii) $\tau(u^*u) \geq 0$, $u \in \mathfrak{S}$, and $\tau(u^*u) = 0$ only if $u = 0$.

The trace τ yields an inner product $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathfrak{S}$, where $\langle\, v \,|\, u \,\rangle = \tau(u^*v)$, $u,\ v\ \in \mathfrak{S}$. Let $\mathfrak{S}^\dagger = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{S},\mathbb{R})$ be the dual space of $\mathfrak{S}$, assign to each $u \in \mathfrak{S}$ the linear functional $\langle\,\cdot\,|\,u\,\rangle \in \mathfrak{S}^\dagger$ given by $\langle\,\cdot\,|\,u\,\rangle(v) = \langle\,v\,|\,u\,\rangle$, $v \in \mathfrak{S}$, and get a vector space isomorphism from $\mathfrak{S}$ to $\mathfrak{S}^\dagger$. Accordingly, we may introduce a multiplication operation, an involution, and a trace on $\mathfrak{S}^\dagger$ defined as

$$\langle\,\cdot\,|\,u\,\rangle\,\langle\,\cdot\,|\,v\,\rangle = \langle\,\cdot\,|\,uv\,\rangle,\ \ \langle\,\cdot\,|\,u\,\rangle^* = \langle\,\cdot\,|\,u^*\,\rangle,\ \ \tau(\langle\,\cdot\,|\,u\,\rangle) = \tau(u),\ \ u,\ v \in \mathfrak{S},$$

and make $\mathfrak{S}^\dagger$ an algebra with a trace isomorphic to $\mathfrak{S}$. With regard to our purposes, we need to impose additional requirements. A particular one is concerned with the existence of an elementary linear basis $\mathfrak{u}[\,\mathfrak{S}\,] = \{\,u_I : I \in \mathcal{I}[\,\mathfrak{S}\,]\,\}$ for $\mathfrak{S}$. The basis $\mathfrak{u}[\,\mathfrak{S}\,]$ is standard, and the attribute elementary refers to the special forms of the structure equations satisfied by the basis elements $u_I \in \mathfrak{S}$, $I \in \mathcal{I}[\,\mathfrak{S}\,]$. We will be also introducing a second multiplication operation on $\mathfrak{S}$ and $\mathfrak{S}^\dagger$, referred to as the normalized product and denoted by ×. The following case studies would make the points.

Case Study I – Matrix Algebras

Let $\mathfrak{M}(n)$ denote the algebra of $n \times n$ real matrices, which is just the algebra of linear operators on Euclidean space $\mathbb{R}^n$, $n \geq 2$, and for this reason should be included in the class of geometric algebras. The standard trace $\tau : \mathfrak{M}(n) \to \mathbb{R}$ assigns to each $u = [\lambda_{ij}]_{i,j=1}^n \in \mathfrak{M}(n)$ the sum of its diagonal entries, i.e.,

$$\tau(u) = \lambda_{11} + \lambda_{22} + \cdots + \lambda_{nn}.$$

The algebra $\mathfrak{M}(n)$ has a linear basis $\mathfrak{u}[\,\mathfrak{M}(n)\,] = \{u_{ij} : 1 \leq i, j \leq n\}$ consisting of matrix units, where u_{ij} is the elementary matrix defined by assuming that the (i, j) entry equals 1 and all the other entries are equal to 0. The matrix units are subject to the structure equations

$$1_{\mathfrak{M}(n)} = u_{11} + u_{22} + \cdots + u_{nn}, \tag{3.1}$$

$$u_{ij}\, u_{kl} = \delta(j, k)\, u_{il}, \quad 1 \leq i, j, k, l \leq n, \tag{3.2}$$

$$u_{ij}^* = u_{ji}, \quad 1 \leq i, j \leq n, \tag{3.3}$$

where $\delta(j, k)$ equals 1 or 0 according as $j = k$ or $j \neq k$. The basis $\mathfrak{u}[\,\mathfrak{M}(n)\,]$ is orthonormal relative to the inner product associated with the standard trace. The normalized multiplication operation × on $\mathfrak{M}(n)$ is given by

$$u \times v = n^{-1}\, u\, v, \quad u,\ v \in \mathfrak{M}(n). \tag{3.4}$$

Case Study II – Group with Two–Cocycle Algebras

Suppose G be a finite group of order $|G| \geq 2$, and let $e \in G$ denote its identity. Let $\mathrm{L}^2(G)$ be the Hilbert space of all functions $\xi : G \to \mathbb{R}$ with the inner product

$$\langle \xi, \eta \rangle = \sum_{g \in G} \xi(g)\, \eta(g), \quad \xi,\ \eta \in \mathrm{L}^2(G).$$

The functions $\xi_g : G \to \mathbb{R}$, $\xi_g(h) = \delta(g, h)$, $g, h \in G$, where $\delta(g, h)$ equals 1 or 0 according as $g = h$ or $g \neq h$, form an orthonormal basis for $\mathrm{L}^2(G)$. Consider next the algebra $\mathfrak{L}[\,\mathrm{L}^2(G)\,]$ of linear operators $u : \mathrm{L}^2(G) \to \mathrm{L}^2(G)$, with standard trace $\tau : \mathfrak{L}[\,\mathrm{L}^2(G)\,] \to \mathbb{R}$ given by

$$\tau(u) = |G|^{-1} \sum_{g \in G} \langle u(\xi_g), \xi_g \rangle, \quad u \in \mathfrak{L}[\,\mathrm{L}^2(G)\,]. \tag{3.5}$$

The following concepts will play a special part subsequently.

Definition 3.1. *A function $\varepsilon : G \times G \to \{1, -1\}$ is a two–cocycle of G provided*

(i) $\varepsilon(g, e) = \varepsilon(e, g) = 1, \quad g \in G,$

(ii) $\varepsilon(g, h)\, \varepsilon(gh, k) = \varepsilon(g, hk)\, \varepsilon(h, k), \quad g, h, k \in G.$

Definition 3.2. *The group algebra* $\mathfrak{C}(G,\varepsilon)$ *associated with group* G *and a two–cocycle* $\varepsilon : G \times G \to \{1,-1\}$ *is the subalgebra of* $\mathfrak{L}[\,\mathrm{L}^2(G)\,]$ *generated by the set* $\mathfrak{u}[\,\mathfrak{C}(G,\varepsilon)\,] = \{\, u_g : g \in G\,\} \subseteq \mathfrak{L}[\,\mathrm{L}^2(G)\,]$, *where*

$$u_g : \mathrm{L}^2(G) \to \mathrm{L}^2(G), \quad u_g(\xi_k) = \varepsilon(g,k)\,\xi_{gk}, \quad g,\, k \in G. \tag{3.6}$$

The operators u_g, $g \in G$, are unitary and satisfy the structure equations

$$1_{\mathfrak{C}(G,\,\varepsilon)} = u_e, \tag{3.7}$$

$$u_g\, u_h = \varepsilon(g,h)\, u_{gh}, \quad g, h \in G, \tag{3.8}$$

$$u_g^* = \varepsilon(g,g^{-1})\, u_{g^{-1}}, \quad g \in G. \tag{3.9}$$

Equation (3.7) is a consequence of (3.6) and property (i) in Definition 3.1. To prove (3.8) we observe that

$$u_g\, u_h(\xi_k) = \varepsilon(h,k)\, u_g(\xi_{hk}) = \varepsilon(h,k)\,\varepsilon(g,hk)\,\xi_{ghk}, \quad g,h,k \in G,$$

$$\varepsilon(g,h)\, u_{gh}(\xi_k) = \varepsilon(g,h)\,\varepsilon(gh,k)\,\xi_{ghk}, \quad g,h,k \in G,$$

and then rely on property (ii) in Definition 3.1. Equation (3.9) follows from $u_g^* = u_g^{-1}$ in conjunction with $u_g^{-1} = \varepsilon(g,g^{-1})\, u_{g^{-1}}$, $g \in G$. Moreover, properties (i) and (ii) in Definition 3.1 are equivalent to equations (3.7), (3.8), respectively. Using (3.5) and (3.6) we get

$$\tau(u_g) = |G|^{-1} \sum_{h\in G} \langle u_g(\xi_h), \xi_h\rangle = |G|^{-1} \sum_{h\in G} \varepsilon(g,h)\,\langle \xi_{gh}, \xi_h\rangle$$

$$= |G|^{-1} \sum_{h\in G} \varepsilon(g,h)\,\delta(gh,h), \quad g \in G,$$

hence $\tau(u_e) = 1$ and $\tau(u_g) = 0$, $g \neq e$. Relative to the inner product $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathfrak{C}(G,\varepsilon)$ associated with the faithful trace τ we observe that

$$\langle\, u_h \,|\, u_g \,\rangle = \tau(u_g^* u_h) = \varepsilon(g,g^{-1})\,\tau(u_{g^{-1}} u_h) = \varepsilon(g,g^{-1})\,\varepsilon(g^{-1},h)\,\tau(u_{g^{-1}h}),$$

a set of equations which shows that the basis $\mathfrak{u}[\,\mathfrak{C}(G,\varepsilon)\,]$ is orthonormal.

The normalized multiplication operation $\times$ on $\mathfrak{C}(G,\varepsilon)$ is given by

$$u \times v = |G|^{-1}\, u\, v, \quad u,\ v \in \mathfrak{C}(G,\varepsilon). \tag{3.10}$$

To make a point, we note the similarities between the equations (3.1) – (3.4) for algebra $\mathfrak{M}(n)$ and the equations (3.7) – (3.10) for algebra $\mathfrak{C}(G,\varepsilon)$.

Case Study III – Clifford Algebras

Clifford algebras form a particular collection of group with two–cocycle algebras that deserve special attention due to their use in Spin geometry and Clifford analysis. The real Clifford algebra $\mathfrak{C}_{n,m}(\mathbb{R})$, $n, m \geq 0$, of signature (n,m) is defined by assuming that $\mathbb{R} \oplus \mathbb{R}^{n+m} \subseteq$

$\mathfrak{C}_{n,m}(\mathbb{R})$, $1_{\mathfrak{C}_{n,m}(\mathbb{R})} = 1 \in \mathbb{R}$, and the orthonormal basis $\{e_1, e_2, \ldots, e_{n+m}\}$ for $\mathbb{R}^{n+m}$ is a complete set of generators satisfying the Clifford relations

$$e_i\, e_j + e_j\, e_i = 0, \quad 1 \le i, j \le n+m,\ i \ne j,$$

$$e_k^2 = -1,\ 1 \le k \le n, \quad e_k^2 = 1,\ n+1 \le k \le n+m.$$

In particular, $\mathfrak{C}_{0,0}(\mathbb{R}) = \mathbb{R}$. Assuming that $(n,m) \ne (0,0)$ and $0 \le p \le n+m$, we let $\mathcal{I}^p_{n+m}$ denote the collection of all p–element subsets $I \subseteq \{1, 2, \ldots, n+m\}$. If $p = 0$, $\mathcal{I}^0_{n+m} = \{\emptyset\}$. Each $I \in \mathcal{I}^p_{n+m}$, $p \ge 1$, is expressed as a p–tuple

$$I = (i_1, i_2, \ldots, i_p), \quad 1 \le i_1 < i_2 < \cdots < i_p \le n+m.$$

We next form the union $\mathcal{I}_{n+m} = \bigcup_{p=0}^{n+m} \mathcal{I}^p_{n+m}$ and assign to every $I \in \mathcal{I}_{n+m}$ the element $u_I \in \mathfrak{C}_{n,m}(\mathbb{R})$ given by

$$u_\emptyset = 1,\ u_I = e_{i_1} e_{i_2} \cdots e_{i_p}, \quad I = (i_1, i_2, \ldots, i_p), \quad 1 \le p \le n+m.$$

Since these are all possible reduced products of generators of $\mathfrak{C}_{n,m}(\mathbb{R})$, the set $\{u_I : I \in \mathcal{I}_{n+m}\}$ is a linear basis for $\mathfrak{C}_{n,m}(\mathbb{R})$. Operation Δ of symmetric difference introduces a group structure on $\mathcal{I}_{n+m}$, and the Clifford relations yield a function $\varepsilon_{n,m} : \mathcal{I}_{n+m} \times \mathcal{I}_{n+m} \to \{1, -1\}$ uniquely determined by the equations $u_I\, u_J = \varepsilon_{n,m}(I,J)\, u_{I \Delta J}$, $I, J \in \mathcal{I}_{n+m}$. Function $\varepsilon_{n,m}$ is a two–cocycle on $\mathcal{I}_{n+m}$, hence $\mathfrak{C}_{n,m}(\mathbb{R})$ is the group algebra $\mathfrak{C}(\mathcal{I}_{n+m}, \varepsilon_{n,m})$. The algebras $\mathfrak{C}_{n,0}(\mathbb{R})$, $n \ge 1$, are called Euclidean Clifford algebras, and the first three such algebras are $\mathfrak{C}_{1,0}(\mathbb{R}) = \mathbb{C}$, the complex numbers, $\mathfrak{C}_{2,0}(\mathbb{R}) = \mathbb{H}$, the Hamilton quaternions, and $\mathfrak{C}_{3,0}(\mathbb{R}) = \mathbb{H} \oplus \mathbb{H}$, the split biquaternions.

We end this subsection with the definition of geometric algebras.

Definition 3.3. *A unital involutive algebra* $\mathfrak{G}$ *is a geometric algebra provided*

$$\mathfrak{G} = \mathfrak{G}_1 \oplus \mathfrak{G}_2 \oplus \cdots \oplus \mathfrak{G}_\kappa, \tag{3.11}$$

where each $\mathfrak{G}_\iota$, $1 \le \iota \le \kappa$, *is either a matrix algebra* $\mathfrak{M}(n_\iota)$, $n_\iota \ge 2$, *or a group with two–cocycle algebra* $\mathfrak{C}(G_\iota, \varepsilon_\iota)$, $|G_\iota| \ge 2$.

Algebra $\mathfrak{G}$ with the direct sum decomposition (3.11) has a standard faithful trace $\tau : \mathfrak{G} \to \mathbb{R}$, and an elementary linear basis $\mathfrak{u}[\,\mathfrak{G}\,]$ which is just the collection of all elementary bases $\mathfrak{u}[\,\mathfrak{G}_\iota\,]$, $1 \le \iota \le \kappa$. Using the specific normalized multiplication operations $\times$ on each componenent $\mathfrak{G}_\iota$, $1 \le \iota \le \kappa$, given $u, v \in \mathfrak{G}$ expressed as

$$u = u_1 + u_2 + \cdots + u_\kappa,\ v = v_1 + v_2 + \cdots + v_\kappa,\ u_\iota,\, v_\iota \in \mathfrak{G}_\iota,\ 1 \le \iota \le \kappa,$$

the normalized product $u \times v \in \mathfrak{G}$ is defined by

$$u \times v = u_1 \times v_1 + u_2 \times v_2 + \cdots + u_\kappa \times v_\kappa. \tag{3.12}$$

3.2 Hom Environments — Structure Manifolds

Suppose $\mathfrak{G}$ is a geometric algebra as in Definition 3.3, with trace τ, inner product $\langle \cdot \,|\, \cdot \rangle$, and elementary basis $\mathfrak{u}[\,\mathfrak{G}\,] = \{\, u_I : I \in \mathcal{I}[\,\mathfrak{G}\,]\,\}$. Let $\mathfrak{A}$ be a unital involutive algebra. The tensor products $\mathfrak{G} \otimes_{\mathbb{R}} \mathfrak{A}$ and $\mathfrak{G}^{\dagger} \otimes_{\mathbb{R}} \mathfrak{A} \equiv \mathrm{Hom}_{\mathbb{R}}(\mathfrak{G}, \mathfrak{A})$, where $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{G}, \mathfrak{A})$ is the space of linear mappings from $\mathfrak{G}$ to $\mathfrak{A}$, are isomorphic $\mathfrak{A}$–bimodules with bases $\{\, u_I \otimes 1_{\mathfrak{A}} : I \in \mathcal{I}[\,\mathfrak{G}\,]\,\}$ and $\{\, \langle \cdot \,|\, u_I \,\rangle\, 1_{\mathfrak{A}} : I \in \mathcal{I}[\,\mathfrak{G}\,]\,\}$. The isomorphism maps $u \otimes a \in \mathfrak{G} \otimes_{\mathbb{R}} \mathfrak{A}$ to $\langle \cdot \,|\, u \,\rangle\, a \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{G}, \mathfrak{A})$, $u \in \mathfrak{G}$, $a \in \mathfrak{A}$.

Definition 3.4. *The Hom environment $\mathfrak{H}[\,\mathfrak{G}, \mathfrak{A}\,]$ associated with a geometric algebra $\mathfrak{G}$ and a unital involutive algebra $\mathfrak{A}$ is defined by assuming that*

(i) *the base algebra of $\mathfrak{H}[\,\mathfrak{G}, \mathfrak{A}\,]$ equals $\mathfrak{A}$,*

(ii) *the total algebra of $\mathfrak{H}[\,\mathfrak{G}, \mathfrak{A}\,]$ equals the $\mathfrak{A}$–bimodule $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{G}, \mathfrak{A})$ with the normalized multiplication operation*

$$\langle \cdot \,|\, u \,\rangle\, a \times \langle \cdot \,|\, v \,\rangle\, b = \langle \cdot \,|\, u \times v \,\rangle\, ab, \quad u,\, v \in \mathfrak{G},\;\; a,\, b \in \mathfrak{A}, \tag{3.13}$$

and the involution operation

$$(\langle \cdot \,|\, u \,\rangle\, a)^* = \langle \cdot \,|\, u^* \,\rangle\, a^*,\; u \in \mathfrak{G},\;\; a \in \mathfrak{A}. \tag{3.14}$$

(iii) *the environment projection $\Pi : \mathrm{Hom}_{\mathbb{R}}(\mathfrak{G}, \mathfrak{A}) \to \mathfrak{A}$ is given by*

$$\Pi(\langle \cdot \,|\, u \,\rangle\, a) = \tau(u)\, a, \quad u \in \mathfrak{G},\;\; a \in \mathfrak{A}. \tag{3.15}$$

The total algebra $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{G}, \mathfrak{A})$ of environment $\mathfrak{H}[\,\mathfrak{G}, \mathfrak{A}\,]$ includes the structure manifolds $\mathcal{S}(\mathfrak{H}[\,\mathfrak{G}, \mathfrak{A}\,])$, $S_*(\mathfrak{H}[\,\mathfrak{G}, \mathfrak{A}\,])$ of $\mathfrak{H}[\,\mathfrak{G}, \mathfrak{A}\,]$, and the spaces $\mathrm{Hom}_{\mathrm{alg}}(\mathfrak{G}, \mathfrak{A})$, $\mathrm{Hom}_{\mathrm{alg},*}(\mathfrak{G}, \mathfrak{A})$ of algebra homomorphisms, or involutive homomorphisms, from $\mathfrak{G}$ to $\mathfrak{A}$. Our subsequent goal is to prove the following result.

Theorem C – Spaces of Homomorphisms as Structure Manifolds

Suppose $\mathfrak{H}[\,\mathfrak{G}, \mathfrak{A}\,]$ is a Hom environment with $\mathfrak{G}$ a geometric algebra. Then

(i) $\mathcal{S}(\mathfrak{H}[\,\mathfrak{G}, \mathfrak{A}\,]) = \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{G}, \mathfrak{A})$,

(ii) $\mathcal{S}_*(\mathfrak{H}[\,\mathfrak{G}, \mathfrak{A}\,]) = \mathrm{Hom}_{\mathrm{alg},*}(\mathfrak{G}, \mathfrak{A})$.

The proof will be divided into three lemmas. The first two are concerned with the cases when $\mathfrak{G}$ is a matrix or a group algebra.

Lemma 3.5. *Theorem* C *for* $\mathfrak{G} = \mathfrak{M}(n)$, $n \geq 2$.

Proof. Let $\mathfrak{u}[\,\mathfrak{M}(n)\,] = \{u_{ij} : 1 \leq i, j \leq n\}$ be the elementary basis of $\mathfrak{M}(n)$. Suppose $\alpha \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{M}(n), \mathfrak{A})$ and let $\mathfrak{a}(\alpha) = \{a_{ij} \in \mathfrak{A} : 1 \leq i, j \leq n\}$ denote the corresponding coefficient set of α, i.e., assume that

$$\alpha = \sum_{1 \leq i,j \leq n} \langle \cdot \,|\, u_{ij} \,\rangle\, a_{ij}. \tag{3.16}$$

We will only prove part (ii) in Theorem C. Part (i) follows by disregarding the involution operations on $\mathfrak{M}(n)$ and $\mathfrak{A}$. The proof reduces to showing the equivalence of the next three statements.

(i) $\alpha \in \mathrm{Hom}_{\mathrm{alg},*}(\mathfrak{M}(n), \mathfrak{A})$,

(ii) The coefficient set $\mathfrak{a}(\alpha) = \{a_{ij} \in \mathfrak{A} : 1 \leq i, j \leq n\}$ satisfies the equations

$$1_{\mathfrak{A}} = a_{11} + a_{22} + \cdots + a_{nn}, \tag{3.17}$$

$$a_{ij}\, a_{kl} = \delta(j,k)\, a_{il}, \quad 1 \leq i,j,k,l \leq n, \tag{3.18}$$

$$a^*_{ij} = a_{ji}, \quad 1 \leq i,j \leq n, \tag{3.19}$$

(iii) $\alpha \in \mathcal{S}_*(\mathfrak{H}[\,\mathfrak{M}(n,\mathfrak{A}\,])$.

The equivalence between (i) and (ii) is obvious. Since $\alpha(u_{ij}) = a_{ij}$, $1 \leq i, j \leq n$, we note that equations (3.17)–(3.19) and the structure equations (3.1)–(3.3) are consistent with, and lead to, $\alpha \in \mathrm{Hom}_{\mathrm{alg},*}(\mathfrak{M}(n), \mathfrak{A})$.

To get statement (iii) from (ii) we need to prove that α satisfies requirements (i)–(vi) in Definition 2.5. Observe first that

$$\Pi(\alpha) = \sum_{1\leq i,j\leq n} \tau(u_{ij})\, a_{ij} = \sum_{1\leq i,j\leq n} \delta(i,j)\, a_{ij} = \sum_{1\leq i\leq n} a_{ii},$$

hence requirement (i), i.e., $\Pi(\alpha) = 1_{\mathfrak{A}}$, is equivalent to equation (3.17). Next, using (3.16), (3.4), (3.2), (3.18) we get

$$\begin{aligned}\alpha \times \alpha &= \sum_{1\leq i,j,k,l\leq n} \langle\, \cdot \,|\, u_{ij} \times u_{kl}\,\rangle\, a_{ij} a_{kl} = \sum_{1\leq i,j,k,l\leq n} n^{-1} \langle\, \cdot \,|\, u_{ij} u_{kl}\,\rangle\, a_{ij} a_{kl} \\ &= \sum_{1\leq i,j,k,l\leq n} n^{-1}\, \delta(j,k)^2 \langle\, \cdot \,|\, u_{il}\,\rangle\, a_{il} = \sum_{1\leq i,l\leq n} \langle\, \cdot \,|\, u_{il}\,\rangle\, a_{il} = \alpha,\end{aligned} \tag{3.20}$$

which is requirement (ii). To check requirements (iii), (iv), i.e., to show that

$$\Pi(\varphi \times \alpha)\cdot \alpha = \varphi \times \alpha, \quad \alpha \cdot \Pi(\alpha \times \psi) = \alpha \times \psi, \quad \varphi,\, \psi \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{M}(n), \mathfrak{A}), \tag{3.21}$$

we introduce the basis $\{\delta_{ij} = n\,\langle\, \cdot \,|\, u^*_{ij}\rangle\, 1_{\mathfrak{A}} : 1 \leq i, j \leq n\}$ for $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{M}(n), \mathfrak{A})$ as an $\mathfrak{A}$–bimodule, and note that it would be enough to prove (3.21) when $\varphi = \delta_{ij}$ and $\psi = \delta_{kl}$. Using previous definitions and equations we get

$$\begin{aligned}\delta_{ij} \times \alpha &= \sum_{1\leq k,l\leq n} n\,\langle\, \cdot \,|\, u^*_{ij} \times u_{kl}\rangle\, a_{kl} = \sum_{1\leq k,l\leq n} \langle\, \cdot \,|\, u_{ji} u_{kl}\rangle\, a_{kl} \\ &= \sum_{1\leq k,l\leq n} \delta(i,k)\, \langle\, \cdot \,|\, u_{jl}\rangle\, a_{kl} = \sum_{1\leq l\leq n} \langle\, \cdot \,|\, u_{jl}\rangle\, a_{il},\end{aligned}$$

hence

$$\Pi(\delta_{ij} \times \alpha) = \sum_{1\leq l\leq n} \tau(u_{jl})\, a_{il} = \sum_{1\leq l\leq n} \delta(j,l) a_{il} = a_{ij}. \tag{3.22}$$

Therefore, by (3.18) we have

$$\Pi(\delta_{ij} \times \alpha)\, \cdot\, \alpha = \sum_{1\leq k,l\leq n} \langle\, \cdot \,|\, u_{kl}\rangle\, a_{ij} a_{kl} = \sum_{1\leq k,l\leq n} \delta(j,k)\, \langle\, \cdot \,|\, u_{kl}\rangle\, a_{il}$$

$$= \sum_{1\leq l\leq n} \langle \,\cdot\,|\, u_{jl}\rangle\, a_{il} = \delta_{ij} \times \alpha, \tag{3.23}$$

which is requirement (iii) in (3.21) for $\varphi = \delta_{ij}$. Similar calculations imply

$$\Pi(\alpha \times \delta_{kl}) = a_{kl}, \quad \alpha \,\cdot\, \Pi(\alpha \times \delta_{kl}) = \alpha \times \delta_{kl}, \tag{3.24}$$

and part of this is requirement (iv) in (3.21) for $\psi = \delta_{kl}$. Requirement (v), i.e.,

$$\Pi(\varphi \times \alpha)\, \Pi(\alpha \times \psi) = \Pi(\varphi \times \alpha \times \psi), \quad \varphi,\, \psi \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{M}(n), \mathfrak{A}),$$

is a consequence of requirements (iii) and (iv) stated in equation (3.21). Actually, we just need the special forms

$$\Pi\delta_{ij} \times \alpha)\, \Pi(\alpha \times \delta_{kl}) = \Pi\delta_{ij} \times \alpha \times \delta_{kl}), \quad 1 \leq i.\, j,\, k,\, l \leq n,$$

do some more direct calculations that yield the explicit formulas

$$\Pi\delta_{ij} \times \alpha \times \delta_{kl}) = \delta(j,k)\, a_{il}, \quad 1 \leq i.\, j,\, k,\, l \leq n,$$

and then, in conjunction with (3.22), (3.24), observe that requirement (v) is equivalent to property (3.18) of the coefficient set $\mathfrak{a}(\alpha)$.

Finally, requirement (vi), $\alpha^* = \alpha$, is equivalent to property (3.19), as it follows by comparing coefficients in the equations

$$\alpha^* = \sum_{1\leq i,j\leq n} \langle \,\cdot\,|\, u_{ij}^* \,\rangle\, a_{ij}^* = \sum_{1\leq i,j\leq n} \langle \,\cdot\,|\, u_{ji} \,\rangle\, a_{ij}^*, \quad \alpha = \sum_{1\leq i,j\leq n} \langle \,\cdot\,|\, u_{ji} \,\rangle\, a_{ji}.$$

We still need to check that statement (iii) implies (ii), i. e., to derive equations (3.17), (3.18), (3.19) from requirements in Definition 2.5. As a matter of fact, we already proved that the three equations are equivalent to requirements (i), (v), and (vi), respectively. The proof is complete. □

Lemma 3.6. *Theorem* C *for* $\mathfrak{S} = \mathfrak{C}(G,\varepsilon)$, $|G| \geq 2$.

Proof. We let $\mathfrak{u}[\,\mathfrak{C}(G,\varepsilon)\,] = \{u_g : g \in G\}$ be the elementary basis for $\mathfrak{C}(G,\varepsilon)$, and assign to $\alpha \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{C}(G,\varepsilon), \mathfrak{A})$ the set $\mathfrak{a}(\alpha) = \{a_g \in \mathfrak{A} : g \in G\}$ such that

$$\alpha = \sum_{k\in G} \langle \,\cdot\,|\, u_k \,\rangle\, a_k. \tag{3.25}$$

As in the proof of Lemma 3.5, we will get part (ii) in Theorem C by checking that the next three statements are equivalent.

(i) $\alpha \in \mathrm{Hom}_{\mathrm{alg},*}(\mathfrak{C}(G,\varepsilon), \mathfrak{A})$.

(ii) The coefficient set $\mathfrak{a}(\alpha) = \{a_g \in \mathfrak{A} : g \in G\}$ satisfies the equations

$$1_{\mathfrak{A}} = a_e, \tag{3.26}$$

$$a_g\, a_h = \varepsilon(g,h)\, a_{gh}, \quad g,\, h \in G, \tag{3.27}$$

$$a_g^* = \varepsilon(g, g^{-1})\, a_{g^{-1}}, \quad g \in G. \tag{3.28}$$

(iii) $\alpha \in \mathcal{S}_*(\mathfrak{H}[\,\mathfrak{C}(G,\varepsilon), \mathfrak{A}\,])$.

The equivalence of (i) and (ii) is obvious. Since by (3.25) $\alpha(u_g) = a_g$, $g \in G$, statement (i) reduces to the consistency of structure equations (3.7)–(3.9) for $\mathfrak{u}[\,\mathfrak{C}(G,\varepsilon)\,]$ with the properties (3.26)–(3.28) of $\mathfrak{a}(\alpha)$. To get statement (iii) from (ii), we should prove that α satisfies requirements (i)–(vi) in Definition 2.5. Since $\Pi(\alpha) = \sum_{k\in G} \tau(u_k)\, a_k = a_e$, requirement (i) is equivalent to (3.26). Requirement (ii) follows from (3.10), (3.8), (3.27) by observing that

$$\alpha \times \alpha = \sum_{g,h\in G} \langle \cdot \,|\, u_g \,\rangle \times \langle \cdot \,|\, u_h \,\rangle\, a_g\, a_h = |G|^{-1} \sum_{g,h\in G} \langle \cdot \,|\, u_g\, u_h \,\rangle\, a_g\, a_h$$

$$= |G|^{-1} \sum_{k\in G} \sum_{g,h\in G,\, gh=k} \varepsilon(g,h)^2\, \langle \cdot \,|\, u_{gh} \rangle\, a_{gh} = \sum_{k\in G} \langle \cdot \,|\, u_k \,\rangle\, a_k = \alpha.$$

Next, we introduce the basis $\{\delta_g = |G|\, \langle \cdot \,|\, u_g^* \,\rangle\, 1_{\mathfrak{A}} \,:\, g \in G\}$ for $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{C}(G,\varepsilon), \mathfrak{A})$ as an $\mathfrak{A}$–bimodule and consistent with (3.25) and (3.15) get

$$\delta_g \times \alpha = \sum_{k\in G} \langle \cdot \,|\, u_g^*\, u_k \rangle\, a_k, \quad \alpha \times \delta_h = \sum_{k\in G} \langle \cdot \,|\, u_k\, u_h^* \,\rangle\, a_k, \quad g,\, h \in G, \tag{3.29}$$

$$\Pi(\delta_g \times \alpha) = \sum_{k\in G} \tau(u_k^*\, u_g)\, a_k = \sum_{k\in G} \delta(k,g)\, a_k = a_g, \quad g \in G, \tag{3.30}$$

$$\Pi(\alpha \times \delta_h) = \sum_{k\in G} \tau(u_h\, u_k^*)\, a_k = \sum_{k\in G} \delta(h,k)\, a_k = a_h, \quad h \in G. \tag{3.31}$$

Refering to requirements (iii) and (iv) as stated in Definition 2.5, it would be enough to check them when $\varphi = \delta_g$, $\psi = \delta_h$, g, $h \in G$. As expected, we rely on property (3.27) of $\mathfrak{a}(\alpha)$. We start with

$$\Pi(\delta_g \times \alpha) \cdot \alpha = a_g \,\cdot\, \alpha = \sum_{l\in G} \langle \cdot \,|\, u_l \,\rangle\, a_g\, a_l = \sum_{l\in G} \varepsilon(g,l)\, \langle \cdot \,|\, u_l \,\rangle a_{gl},$$

set $l = g^{-1}k$, $k \in G$, hence $\langle \cdot \,|\, u_l \,\rangle = \varepsilon(g^{-1},k)\, \langle \cdot \,|\, u_{g^{-1}} u_k \rangle$, and get

$$\Pi(\delta_g \times \alpha) \,\cdot\, \alpha = \sum_{k\in G} \varepsilon(g, g^{-1}k)\, \varepsilon(g^{-1},k)\, \langle \cdot \,|\, u_{g^{-1}} u_k \rangle\, a_k.$$

Since ε is a two–cocycle, we have

$$\varepsilon(g, g^{-1}k)\, \varepsilon(g^{-1},k)\, u_{g^{-1}} = \varepsilon(g, g^{-1})\, \varepsilon(e,k)\, u_{g^{-1}} = \varepsilon(g, g^{-1})\, u_{g^{-1}} = u_g^*,$$

and after some substitutions conclude that $\Pi(\delta_g \times \alpha) \cdot \alpha = \delta_g \times \alpha$, $g \in G$, which is requirement (iii). Similar calculations show that requirement (iv) is true as well, i.e., $\alpha \cdot \Pi(\alpha \times \delta_h) = \alpha \times \delta_h$, $h \in G$. Requirement (v) is a consequence of (iii) and (iv). Its special forms,

$\Pi(\delta_g \times \alpha)\,\Pi(\alpha \times \delta_h) = \Pi(\delta_g \times \alpha \times \delta_h)$, $g,\, h \in G$, combined with (3.30), (3.31), and the next easy to check formulas

$$\Pi(\delta_g \times \alpha \times \delta_h) = \sum_{k \in G} \varepsilon(g,h)\,\delta(gh,k)\,a_k = \varepsilon(g,h)\,a_{gh}, \quad g,\, h \in G, \tag{3.32}$$

which are true for any $\alpha \in \operatorname{Hom}_{\mathbb{R}}(\mathfrak{C}(G,\varepsilon),\mathfrak{A})$, imply that requirement (v) is equivalent to property (3.27). With regard to requirement (vi), i.e., $\alpha^* = \alpha$, we note that

$$\alpha^* = \sum_{g \in G} \langle\,\cdot\,|\,u_g^*\rangle\,a_g^* = \sum_{g \in G} \varepsilon(g,g^{-1})\,\langle\,\cdot\,|\,u_{g^{-1}}\rangle\,a_g^*, \quad \alpha = \sum_{g \in G} \langle\,\cdot\,|\,u_{g^{-1}}\rangle\,a_{g^{-1}},$$

and then conclude that requirement (vi) and property (3.28) are equivalent.

Finally, we get statement (iii) from (ii) because, by previous remarks, properties (3.26), (3.27), (3.28) of $\mathfrak{a}(\alpha)$ follow from the requirements (i), (v), (vi), which are satisfied by $\alpha \in \mathcal{S}_*(\mathfrak{H}[\,\mathfrak{C}(G,\varepsilon),\mathfrak{A}\,])$, respectively. The proof is complete. ☐

Lemma 3.7. *Theorem* C *for* $\mathfrak{S} = \mathfrak{S}_1 \oplus \mathfrak{S}_2 \oplus \cdots \oplus \mathfrak{S}_\kappa$, *where each* $\mathfrak{S}_\iota$, $1 \le \iota \le \kappa$, *is either a matrix algebra* $\mathfrak{M}(n_\iota)$, $n_\iota \ge 2$, *or a group with two–cocycle algebra* $\mathfrak{C}(G_\iota,\varepsilon_\iota)$, $|G_\iota| \ge 2$.

Proof. We let $\mathfrak{u}(\mathfrak{S}_\iota) = \{\, u_{I,\iota} : I \in \mathcal{I}(\mathfrak{S}_\iota)\,\}$ be the elementary bases for algebras $\mathfrak{S}_\iota$, $1 \le \iota \le \kappa$, introduce the elementary basis $\mathfrak{u}(\mathfrak{S}) = \bigcup_{1 \le \iota \le \kappa} \mathfrak{u}(\mathfrak{S}_\iota)$ for $\mathfrak{S}$, and set up the structure equations, $\operatorname{Eq}[\,\mathfrak{u}(\mathfrak{S})\,]$, for $\mathfrak{S}$. The list splits into three parts.

$\operatorname{Eq}_{\mathrm{unit}}[\,\mathfrak{u}(\mathfrak{S})\,]$ – *Unit Rule.* Denote by $\mathcal{I}_0(\mathfrak{S}_\iota) \subseteq \mathcal{I}(\mathfrak{S}_\iota)$, $1 \le \iota \le \kappa$, the sets used to express each $1_{\mathfrak{S}_\iota}$ in terms of basis elements and record the single equation

$$1_{\mathfrak{S}} = \sum_{1 \le \iota \le \kappa} \sum_{I \in \mathcal{I}_0(\mathfrak{S}_\iota)} u_{I,\iota}.$$

$\operatorname{Eq}_{\mathrm{mult}}[\,\mathfrak{u}(\mathfrak{S})\,]$ – *Multiplication Rules.* Include the rules for products involving basis elements of each $\mathfrak{u}(\mathfrak{S}_\iota)$, $1 \le \iota \le \kappa$, and the additional equations

$$u_{I,\iota}\, u_{J,\varkappa} = 0, \quad I \in \mathcal{I}(\mathfrak{S}_\iota),\ \ J \in \mathcal{I}(\mathfrak{S}_\varkappa),\ \ 1 \le \iota,\, \varkappa \le \kappa,\ \ \iota \ne \varkappa,$$

$\operatorname{Eq}_{\mathrm{inv}}[\,\mathfrak{u}(\mathfrak{S})\,]$ – *Involution Rules.* Rules for every $u_{I,\iota}^*$, $1 \le \iota \le \kappa$, $I \in \mathcal{I}(\mathfrak{S}_\iota)$.

Consistent with the direct sum decomposition, algebra $\mathfrak{S}$ inherits a standard trace τ and an inner product $\langle\,\cdot\,|\,\cdot\,\rangle$. Each element $\alpha \in \operatorname{Hom}_{\mathbb{R}}(\mathfrak{S},\mathfrak{A})$ has a unique coefficient set $\mathfrak{a}(\alpha) = \{a_{I,\iota} \in \mathfrak{A} : 1 \le \iota \le \kappa,\ I \in \mathcal{I}(\mathfrak{S}_\iota)\}$ such that

$$\alpha = \sum_{1 \le \iota \le \kappa} \sum_{I \in \mathcal{I}(\mathfrak{S}_\iota)} \langle\,\cdot\,|\,u_{I,\iota}\,\rangle\,a_{I,\iota}.$$

To prove part (ii) and implicitly part (i) in Theorem C, we have to show that the following three statements are equivalent.

(i) $\alpha \in \operatorname{Hom}_{\mathrm{alg},*}(\mathfrak{S},\, \mathfrak{A})$.

(ii) Set $\mathfrak{a}(\alpha)$ satisfies the collection of equations $\operatorname{Eq}[\,\mathfrak{u}(\mathfrak{a})\,]$ derived by substituting the coefficients $a_{I,\iota}$ for $u_{I,\iota}$ into each structure equation from $\operatorname{Eq}[\,\mathfrak{u}(\mathfrak{S})\,]$.

(iii) $\alpha \in \mathcal{S}_*(\mathfrak{H}[\,S,\mathfrak{A}\,])$.

We rely on Lemmas 3.5, 3.6 and proceed as in their proofs. Statements (i) and (ii) are equivalent, the assumptions in statement (ii) imply that $\alpha \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{S}, \mathfrak{A})$ is subject to the requirements (i)–(vi) in Definition 2.5, i.e., $\alpha \in \mathcal{S}_*(\mathfrak{H}[\,S, \mathfrak{A}\,])$, and the parts $\mathrm{Eq}_{\mathrm{unit}}[\,\mathfrak{u}(\mathfrak{a})\,]$, $\mathrm{Eq}_{\mathrm{mult}}[\,\mathfrak{u}(\mathfrak{a})\,]$, $\mathrm{Eq}_{\mathrm{inv}}[\,\mathfrak{u}(\mathfrak{a})\,]$ of $\mathrm{Eq}[\,\mathfrak{u}(\mathfrak{a})\,]$ turn out to be equivalent to requirements (i), (v), (vi) for $\alpha \in \mathcal{S}_*(\mathfrak{H}[\,S, \mathfrak{A}\,])$, respectively. The proof of Theorem C is complete. □

A quick inspection of the proof shows that the structure manifolds $\mathcal{S}(\mathfrak{H}[\,\mathfrak{S}, \mathfrak{A}\,])$ and $\mathcal{S}_*(\mathfrak{H}[\,\mathfrak{S}, \mathfrak{A}\,])$ of Hom environments are completely defined by requirements (i), (v), or (i), (v), (vi), a comment stated earlier with regard to Definition 2.5.

3.3 Hom Environments — Zariski Tangent Spaces

We continue assuming that $\mathfrak{S}$ is a geometric algebra and $\mathfrak{A}$ a unital involutive algebra. By Theorem C, $\mathrm{Hom}_{\mathrm{alg}}(\mathfrak{S}, \mathfrak{A})$ and $\mathrm{Hom}_{\mathrm{alg},*}(\mathfrak{S}, \mathfrak{A})$ are algebraic manifolds. Definition 2.6 that uses the algebra of dual numbers $\Lambda^{\#}(\mathbb{R}) \equiv \mathbb{R}[\delta]$ has a natural counterpart. Suppose $\alpha \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{S}, \mathfrak{A})$, i.e.,

$$\alpha(1_{\mathfrak{S}}) = 1_{\mathfrak{A}}, \quad \alpha(u\,v) = \alpha(u)\,\alpha(v), \quad u,\, v \in \mathfrak{S}, \tag{3.33}$$

form the augmented algebra $\mathfrak{A}[\delta]$, and define $\theta \in \mathrm{T}_\alpha \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{S}, \mathfrak{A})) \subseteq \mathrm{Hom}_{\mathbb{R}}(\mathfrak{S}, \mathfrak{A})$ by the requirement $\alpha + \theta\,\delta \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{S}, \mathfrak{A}[\delta])$. We end up with the properties

$$\theta(1_{\mathfrak{S}}) = 0,\ \theta(u\,v) = \theta(u)\,\alpha(v) + \alpha(u)\,\theta(v),\ u,\, v \in \mathfrak{S}. \tag{3.34}$$

If $\alpha \in \mathrm{Hom}_{\mathrm{alg},*}(\mathfrak{S}, \mathfrak{A})$, then $\theta \in \mathrm{T}_\alpha \mathrm{Hom}_{\mathrm{alg},*}(\mathfrak{S}, \mathfrak{A})$ provided, in addition, $\theta(u^*) = \theta(u)^*$, $u \in \mathfrak{S}$.

Lemmas 2.9, 2.12, and Theorem A prompt us to implement substitutes for other concepts from Section 2. The Lie algebra $\mathrm{Der}(\mathfrak{A})$ of derivations on $\mathfrak{A}$ will play a particular part subsequently. We assign to each $\alpha \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{S}, \mathfrak{A})$ the operator

$$\Delta_\alpha : \mathrm{Der}(\mathfrak{A}) \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{S}, \mathfrak{A}),\ \Delta_\alpha(\mathcal{D}) = \mathcal{D} \circ \alpha, \quad \mathcal{D} \in \mathrm{Der}(\mathfrak{A}), \tag{3.35}$$

and get the following expected results.

Lemma 3.8. *If* $\alpha \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{S}, \mathfrak{A})$ *or* $\alpha \in \mathrm{Hom}_{\mathrm{alg},*}(\mathfrak{S}, \mathfrak{A})$*, then*

$$\mathrm{T}_\alpha \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{S}, \mathfrak{A}) = \{\, \theta \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{S}, \mathfrak{A}) : \theta = \Delta_\alpha(\mathcal{D}),\ \mathcal{D} \in \mathrm{Der}(\mathfrak{A})\,\},$$

$$\mathrm{T}_\alpha \mathrm{Hom}_{\mathrm{alg},*}(\mathfrak{S}, \mathfrak{A}) = \{\, \theta \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{S}, \mathfrak{A}) : \theta = \Delta_\alpha(\mathcal{D}),\ \mathcal{D} \in \mathrm{Der}_*(\mathfrak{A})\,\}.$$

As a consequence, equations (3.34) that characterize tangent vectors could be derived by applying derivations $\mathcal{D} \in \mathrm{Der}(\mathfrak{A})$ to equations (3.34).

We are also going to use the operator $\mathfrak{D} : \mathfrak{A} \to \mathrm{Der}_{\mathrm{inn}}(\mathfrak{A})$ that assigns to $x \in \mathfrak{A}$ the inner derivation $\mathfrak{D}(x) = [\,x,\,\cdot\,]$. Next, associated with $\alpha \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{S}, \mathfrak{A})$ we introduce the symbol operator $\Sigma_\alpha : \mathrm{Hom}_{\mathbb{R}}(\mathfrak{S}, \mathfrak{A}) \to \mathfrak{A}$ using equation (2.3), the adjusted conjugation operator given by

$$\Gamma_\alpha : \mathrm{Der}(\mathfrak{A}) \to \mathrm{Der}(\mathfrak{A}), \quad \Gamma_\alpha = \mathrm{Id}_{\mathrm{Der}(\mathfrak{A})} - 2\ \mathfrak{D} \circ \Sigma_\alpha \circ \Delta_\alpha, \tag{3.36}$$

and define the Lie subalgebra of derivations on $\mathfrak{A}$ compatible with α as

$$\mathrm{Der}_{\alpha,0}(\mathfrak{A}) = \mathrm{Ker}\,\Delta_\alpha = \{\, \mathcal{D} \in \mathrm{Der}(\mathfrak{A}) : \Delta_\alpha(\mathcal{D}) = \mathcal{D} \circ \alpha = 0.\,\}$$

Theorem B and Corollaries 2.16, 2.17 lead to another specific result.

Corollary 3.9. *If* $\alpha \in \mathrm{Hom_{alg}}(\mathfrak{S}, \mathfrak{A})$, *then* $\Gamma_\alpha : \mathrm{Der}(\mathfrak{A}) \to \mathrm{Der}(\mathfrak{A})$ *is an involution, i.e.,* $\Gamma_\alpha^2 = \mathrm{Id}_{\mathrm{Der}(\mathfrak{A})}$, *and its spectral projections* $\Gamma_\alpha^+, \Gamma_\alpha^- : \mathrm{Der}(\mathfrak{A}) \to \mathrm{Der}(\mathfrak{A})$,

$$\Gamma_\alpha^+ = (\,\mathrm{Id}_{\mathrm{Der}(\mathfrak{A})} + \Gamma_\alpha\,)/2, \quad \Gamma_\alpha^- = (\,\mathrm{Id}_{\mathrm{Der}(\mathfrak{A})} - \Gamma_\alpha\,)/2),$$

satisfy $\mathrm{Der}_{\alpha,0}(\mathfrak{A}) = \mathrm{Ran}\,\Gamma_\alpha^+ = \mathrm{Ker}\,\Gamma_\alpha^-$. *Moreover, if* $\theta \in \mathrm{T}_\alpha \mathrm{Hom_{alg}}(\mathfrak{S}, \mathfrak{A})$, *then* $\{\mathcal{D} \in \mathrm{Der}(\mathfrak{A}) : \Delta_\alpha\,(\mathcal{D}) = \theta\,\} = \mathfrak{D} \circ \Sigma_\alpha(\theta) + \mathrm{Der}_{\alpha,0}\,(\mathfrak{A})$. □

4 Geometric Structures on Vector Bundles

Suppose $\mathcal{E} = (E, \pi, M)$ is a smooth vector bundle over a manifold M, with total space E, and projection $\pi : E \to M$. Let $\Gamma^\infty(M, E)$ denote the space of smooth sections of $\mathcal{E}$. Identify the endomorphisms of $\mathcal{E}$ with $C^\infty(M, \mathbb{R})$–linear operators from $\Gamma^\infty(M, E)$ to $\Gamma^\infty(M, E)$, i.e., elements of the algebra $\mathfrak{A}(M, E) = \mathrm{Hom}_{C^\infty(M,\mathbb{R})}\,[\,\Gamma^\infty(M, E),\, \Gamma^\infty(M, E)\,]$. We may assume that bundle $\mathcal{E}$ has an inner product consisting of a smooth family $\langle\,\cdot\,,\cdot\,\rangle_x$ of inner products on the fibers E_x, $x \in M$, and related to it define an involution operation $*$ on algebra $\mathfrak{A}(M, E)$.

Our goal in this section is to describe the derivations and linear connections on vector bundles compatible with, or preserving, prescribed geometric structures.

4.1 Requisites — Derivations and Linear Connections

For convenience, associated with $\mathcal{E} = (E, \pi, M)$ we introduce the larger algebra

$$\mathfrak{L}(M, E) = \mathrm{Hom}_{\mathbb{R}}\,[\,\Gamma^\infty(M, E),\, \Gamma^\infty(M, E)\,]$$

consisting of linear mappings from $\Gamma^\infty(M, E)$ to $\Gamma^\infty(M, E)$. Each function $\lambda \in C^\infty(M, \mathbb{R})$ defines the element $S(\lambda) \in \mathfrak{L}(M, E)$, called a scalar operator, given by $S(\lambda)\,\varphi = \lambda\,\varphi$, $\varphi \in \Gamma^\infty(M, E)$. The space $\mathcal{S}(M, E)$ of scalar operators is a subalgebra of $\mathfrak{A}(M, E)$, that makes it possible to give a purely algebraic description of the spaces $\mathfrak{P}^k(M, E) \subseteq \mathfrak{L}(M, E)$ of linear differential operators $\mathfrak{D} : \Gamma^\infty(M, E) \to \Gamma^\infty(M, E)$ of order $\mathrm{ord}\,(\mathfrak{D}) \leq k$, $k \geq 0$, in an inductive way, by using the commutators $[\,\mathfrak{D}, S(\lambda)\,] = \mathfrak{D}\,S(\lambda) - S(\lambda)\,\mathfrak{D}$. Specifically,

$$\mathfrak{P}^0(M, E) = \{\,\mathfrak{D} \in \mathfrak{L}(M, E) : [\,\mathfrak{D}, S(\lambda)\,] = 0,\, \lambda \in C^\infty(M, \mathbb{R})\},$$

whence $\mathfrak{P}^0(M, E) = \mathfrak{A}(M, E)$, and then, for each $k \geq 0$,

$$\mathfrak{P}^{k+1}(M, E) = \{\,\mathfrak{D} \in \mathfrak{L}(M, E) : [\,\mathfrak{D}, S(\lambda)\,] \in \mathfrak{P}^k(M, E),\, \lambda \in C^\infty(M, \mathbb{R})\}.$$

For more information about the full algebra of differential operators on smooth vector bundles we refer to Narasimhan [46] and Nicolaescu [47].

Definition 4.1. *An operator* $\partial \in \mathfrak{L}(M, E)$ *is called a derivation on* $\mathcal{E} = (E, \pi, M)$ *provided* $[\,\partial, S(\lambda)\,] \in \mathcal{S}(M, E)$ *for any* $\lambda \in C^\infty(M, \mathbb{R})$.

The space of derivations on $\mathcal{E}$, denoted by $\mathfrak{D}(M, E)$, is a $C^\infty(M, \mathbb{R})$–module such that $\mathfrak{P}^0(M, E) \subseteq \mathfrak{D}(M, E) \subseteq \mathfrak{P}^1(M, E)$, and a Lie algebra with Lie product $[\,\partial, \partial'\,] = \partial\,\partial' - \partial'\,\partial$, $\partial, \partial' \in \mathfrak{D}(M, E)$. By Definition 4.1, given $\partial \in \mathfrak{D}(M, E)$ and $\lambda \in C^\infty(M, \mathbb{R})$, there exists a uniquely determined $\delta(\lambda) \in C^{\infty}(M,\mathbb{R})$ such that $[\,\partial, S(\lambda)\,] = S(\delta(\lambda))$. The resulting linear mapping, $\delta : C^{\infty}(M,\mathbb{R}) \to C^{\infty}(M,\mathbb{R})$, turns out to be a derivation on algebra $C^{\infty}(M,\mathbb{R})$, i.e.,

$$\delta(\lambda\,\mu) = \delta(\lambda)\,\mu + \lambda\,\delta(\mu), \quad \lambda,\, \mu \in C^{\infty}(M,\mathbb{R}),$$

and thus we get a $C^{\infty}(M,\mathbb{R})$–linear mapping and Lie algebra homomorphism

$$res : \mathfrak{D}(M,E) \to \mathrm{Der}\,(C^{\infty}(M,\mathbb{R})), \quad res\,\partial = \delta, \quad \partial \in \mathfrak{D}(M,E), \tag{4.1}$$

called the restriction operator.

Definition 4.2. *A $C^{\infty}(M,\mathbb{R})$–linear mapping $\nabla : \mathrm{Der}\,(C^{\infty}(M,\mathbb{R})) \to \mathfrak{D}(M,E)$ is a linear connection on $\mathcal{E} = (E,\pi,M)$ provided $res \circ \nabla = \mathrm{Id}_{\mathrm{Der}\,(C^{\infty}(M,\mathbb{R}))}$.*

Recall that $\mathrm{Der}\,(C^{\infty}(M,\mathbb{R}))$ coincides with $\mathfrak{X}(M) = \Gamma^{\infty}(M,\mathrm{T}M)$, the space of smooth vector fields on M, sections of the tangent vector bundle of M. Therefore, a linear connection ∇ assigns to each $\xi \in \mathfrak{X}(M)$ a derivation $\nabla(\xi)$ referred to as the ∇–covariant derivation on $\mathcal{E} = (E,\pi,M)$ in the direction ξ. In terms of covariant derivations, the requirement in Definition 4.2 is equivalent to

$$\nabla(\xi)(\lambda\,\varphi) = \xi(\lambda)\,\varphi + \lambda\,\nabla(\xi)(\varphi),\ \ \xi \in \mathfrak{X}(M),\ \ \lambda \in C^{\infty}(M,\mathbb{R}),\ \ \varphi \in \Gamma^{\infty}(M,E),$$

which is just the familiar chacterization of linear connections. For definitions and general results concerning linear connections we refer to Helgason [12], Kobayashi, Nomizu [14], and Lang [15]. The next elementary result is a direct consequence of the property $\mathrm{Ker}\,(res) = \mathfrak{A}(M,E)$.

Lemma 4.3. *Let ∂_0 and ∇_0 be a derivation and a linear connection on bundle $\mathcal{E} = (E,\pi,M)$, respectively, both of them arbitrary.*

(i) *The collection of all derivations ∂ on $\mathcal{E}$ such that $res\,\partial = res\,\partial_0$ equals*

$$\partial_0 + \mathfrak{A}(M,E) = \{\,\partial_0 + X : X \in \mathfrak{A}(M,E)\,\}.$$

(ii) *The collection of all linear connections ∇ on $\mathcal{E}$ equals*

$$\nabla_0 + \mathrm{Hom}_{C^{\infty}(M,\mathbb{R})}\,[\,\mathfrak{X}(M),\, \mathfrak{A}(M,E)\,],$$

where $\mathrm{Hom}_{C^{\infty}(M,\mathbb{R})}\,[\,\mathfrak{X}(M),\, \mathfrak{A}(M,E)\,]$ denotes the space of $C^{\infty}(M,\mathbb{R})$–linear mappings from $\mathfrak{X}(M)$ to $\mathfrak{A}(M,E)$, i.e., each linear connection ∇ on $\mathcal{E}$ is represented as $\nabla = \nabla_0 + \Xi$, $\Xi \in \mathrm{Hom}_{C^{\infty}(M,\mathbb{R})}\,[\,\mathfrak{X}(M),\, \mathfrak{A}(M,E)\,]$. □

We proceed with another construction. For any $\partial \in \mathfrak{D}(M,E)$, the commutator identity in algebra $\mathfrak{L}(M,E)$, $[\,\partial, X\,Y\,] = [\,\partial, X\,]\,Y + X\,[\,\partial, Y\,]$, $X,\, Y \in \mathfrak{A}(M,E)$, implies that operator $\mathcal{D} : \mathfrak{A}(M,E) \to \mathfrak{A}(M,E)$, $\mathcal{D}\,(X) = [\,\partial, X\,]$, $X \in \mathfrak{A}(M,E)$, is a derivation on the endomorphism algebra $\mathfrak{A}(M,E)$. The mapping

$$ext : \mathfrak{D}(M,E) \to \mathrm{Der}\,(\mathfrak{A}(M,E)), \quad ext\,\partial = \mathcal{D}, \quad \partial \in \mathfrak{D}(M,E), \tag{4.2}$$

referred to as the extension operator, is $C^{\infty}(M,\mathbb{R})$–linear and a Lie algebra homomorphisms. If $\partial \in \mathfrak{D}(M,E)$, operators res and ext satisfy the identity

$$ext\,\partial\,(\lambda\,X) = res\,\partial\,(\lambda)\,X + \lambda\,ext\,\partial\,(X),\ \ \lambda \in C^{\infty}(M,\mathbb{R}),\ \ X \in \mathfrak{A}(M,E).$$

4.2 Derivations Preserving Geometric Structures

We are now in a position to address the special theme of our article. Throughout this subsection we will assume that $\mathfrak{G}$ is a geometric algebra.

Definition 4.4. *Let $\mathcal{E} = (E, \pi, M)$ be a smooth vector bundle with endomorphism algebra $\mathfrak{A}(M, E)$. Consider the structure manifold $\mathcal{S}\,(\,\mathfrak{H}\,[\,\mathfrak{G}, \mathfrak{A}(M, E)\,]\,)$ of the Hom environment $\mathfrak{H}\,[\,\mathfrak{G}, \mathfrak{A}(M, E)\,]$ and refer to any $\Phi \in \mathcal{S}\,(\,\mathfrak{H}\,[\,\mathfrak{G}, \mathfrak{A}(M, E)\,]\,)$ as an $\mathfrak{G}$–structure on bundle $\mathcal{E}$.*

By Theorem C, we know that each $\mathfrak{G}$–structure Φ on bundle $\mathcal{E}$ is an algebra homomorphism $\Phi \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{G}, \mathfrak{A}(M, E))$. If $\mathcal{E}$ has an inner product, we may require Φ to be self–adjoint, i.e., $\Phi \in \mathrm{Hom}_{\mathrm{alg},*}(\mathfrak{G}, \mathfrak{A}(M, E))$. The existence of $\mathfrak{G}$–structures on vector bundles might be subject to certain intricate obstructions, which we will be taking for granted subsequently. Consistent with definitions and notation from Section 3, we assign to each structure $\Phi \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{G}, \mathfrak{A}(M, E))$ the operator $\Delta_\Phi : \mathrm{Der}(\mathfrak{A}(\mathrm{M}, \mathrm{E})) \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{G}, \mathfrak{A}(\mathrm{M}, \mathrm{E}))$ using equation (3.35). Recall that $\mathrm{Der}_{\Phi,0}\,(\mathfrak{A}(M, E)) = \mathrm{Ker}\,(\Delta_\Phi)$ denotes the subspace of derivations on $\mathfrak{A}(M, E)$ compatible with structure Φ.

Definition 4.5. *Suppose $\Phi \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{G}, \mathfrak{A}(M, E))$ is an $\mathfrak{G}$–structure on vector bundle $\mathcal{E} = (E, \pi, M)$. A derivation $\partial \in \mathfrak{D}(M, E)$ on $\mathcal{E}$ preserves Φ provided $ext\,\partial \in \mathrm{Der}_{\Phi,0}\,(\mathfrak{A}(M, E))$, i.e., $ext\,\partial \circ \Phi = 0$. The subspace of all such derivations ∂ is denoted by $\mathfrak{D}_{\Phi,0}(M, E)$.*

We already defined the concept of linear connections compatible with geometric structures in terms of parallel translation mappings. Standard techniques lead to an equivalent defining property derived from Definitions 4.2, 4.5.

Definition 4.6. *A linear connection ∇ on bundle $\mathcal{E} = (E, \pi, M)$ preserves a structure $\Phi \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{G}, \mathfrak{A}(M, E))$ provided $\nabla(\xi) \in \mathfrak{D}_{\Phi,0}(M, E)$, $\xi \in \mathfrak{X}(M)$.*

Corresponding to $\Phi \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{G}, \mathfrak{A}(M, E))$, in Section 3 we introduced the symbol operator $\Sigma_\Phi : \mathrm{Hom}_{\mathbb{R}}[\,\mathfrak{G}, \mathfrak{A}(\mathrm{M}, \mathrm{E}))\,] \to \mathfrak{A}(\mathrm{M}, \mathrm{E})$ using equation (2.3), and the conjugation operator $\Gamma_\Phi : \mathrm{Der}(\mathfrak{A}(M, E)) \to \mathrm{Der}(\mathfrak{A}(M, E))$ given by equation (3.36). We will rely on both in conjunction with an additional operator.

Definition 4.7. *The conjugation operator $\gamma_\Phi : \mathfrak{D}(M, E) \to \mathfrak{D}(M, E)$ associated with an $\mathfrak{G}$–structure $\Phi \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{G}, \mathfrak{A}(M, E))$ is defined by*

$$\gamma_\Phi = \mathrm{Id}_{\mathfrak{D}(M,E)} - 2\,\Sigma_\Phi \circ \Delta_\Phi \circ ext. \tag{4.3}$$

Operator γ_Φ assigns to each $\partial \in \mathfrak{D}(M, E)$ the mapping $\partial^\dagger = \gamma_\Phi(\,\partial\,) \in \mathfrak{L}(M, E)$. Related to equation (4.3), we define $X_\Phi : \mathfrak{D}(M, E) \to \mathfrak{A}(M, E)$ by

$$X_\Phi(\,\partial\,) = -2\,\Sigma_\Phi \circ \Delta_\Phi \circ ext(\,\partial\,),\;\; \partial \in \mathfrak{D}(M, E), \tag{4.4}$$

get $\partial^\dagger = \partial + X_\Phi(\,\partial\,)$, and then by Lemma 4.3 conclude that $\partial^\dagger \in \mathfrak{D}(M, E)$, as stated in Definition 4.7. Derivation $\partial^\dagger$ is called the Φ–conjugate of ∂.

Theorem D – Structure Preserving as Self–Conjugate Derivations

The operator $\gamma_\Phi : \mathfrak{D}(M, E) \to \mathfrak{D}(M, E)$ has the following properties.

(i) *$res \circ \gamma_\Phi = res$ and $ext \circ \gamma_\Phi = \Gamma_\Phi \circ exp$.*

(ii) γ_Φ *is an involution on* $\mathfrak{D}(M,E)$, *i.e.*, $\gamma_\Phi^2 = \mathrm{Id}_{\mathfrak{D}(M,E)}$.

(iii) $\partial \in \mathfrak{D}_{\Phi,0}(M,E)$ *only if* $\gamma_\Phi(\partial) = \partial$, *i.e.*, $\partial^\dagger = \partial$.

(iv) $\mathfrak{D}_{\Phi,0}(M,E) = \mathrm{Ran}\,(\,\gamma_\Phi^+\,)$, *where* $\gamma_\Phi^+ = (\mathrm{Id}_{\mathfrak{D}(M,E)} + \gamma_\Phi\,)/2$ *is the spectral projection of* γ_Φ *corresponding to the eigenvalue* $\lambda^+ = 1$.

Proof. Suppose $\partial \in \mathfrak{D}(M,E)$, let $\partial^\dagger = \gamma_\Phi(\,\partial\,) = \partial + X_\Phi(\,\partial\,)$, and note that $res(\partial^\dagger) = res(\partial)$ because $X_\Phi(\,\partial\,) \in \mathfrak{A}(M,E) = \mathrm{Ker}\,(res)$. Next, using the operator $\mathfrak{D} : \mathfrak{A}(M,E) \to \mathrm{Der}_{\mathrm{inn}}(\mathfrak{A}(M,E))$, $\mathfrak{D}(X) = [\,X,\,\cdot\,]$, $X \in \mathfrak{A}(M,E)$, and the definition of Γ_Φ, we get that equation (4.4) implies

$$\begin{aligned} ext\,(\partial^\dagger) &= ext\,(\partial) + \mathfrak{D}\,(X_\Phi(\,\partial\,)) = ext\,(\partial) - 2\,\mathfrak{D}\,\circ\,\Sigma_\Phi\,\circ\,\Delta_\Phi\,\circ\,ext\,(\partial) \\ &= [\,\mathrm{Id}_{\mathrm{Der}(\mathfrak{A}(M,E))} - 2\,\mathfrak{D}\,\circ\,\Sigma_\alpha\,\circ\,\Delta_\Phi\,]\,\circ\,ext\,(\partial) = \Gamma_\Phi\,\circ\,ext\,(\partial). \end{aligned}$$

We just proved statement (i). The three other statements follow from Theorem B by making appropriate adjustments. The proof is complete. □

Corollary 4.8. *Let* $\nabla : \mathfrak{X}(M) \to \mathfrak{D}(M,E)$ *be a linear connection on the vector bundle* $\mathcal{E} = (E,\pi,M)$ *with an* $\mathfrak{S}$*–structure* $\Phi \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{S},\mathfrak{A}(M,E))$. *Then, the composite mapping* $\nabla^\dagger = \gamma_\Phi\,\circ\,\nabla$ *is a linear connection on* $\mathcal{E}$ *and* ∇ *preserves structure* Φ *only if* $\nabla^\dagger = \nabla$.

Proof. We rely on Definition 4.2 and observe that

$$res\,(\nabla^\dagger(\xi)) = res\,\circ\,\gamma_\Phi\,(\nabla(\xi)) = res\,(\nabla(\xi)) = \xi, \quad \xi \in \mathfrak{X}(M).$$

Since $\nabla^\dagger(\xi) = \nabla(\xi)^\dagger$, $\xi \in \mathfrak{X}(M)$, Definition 4.6 and part (iii) in Theorem D complete the proof. □

Let $\mathfrak{A}_\Phi(M,E)$ denote the commutator of subalgebra $\mathrm{Ran}\,(\Phi) = \{\,\Phi(u) : u \in \mathfrak{S}\,\}$ in the endomorphism algebra $\mathfrak{A}(M,E)$ defined according to equation (2.4) in Section 2, and let $\pi_\Phi : \mathfrak{A}(M,E) \to \mathfrak{A}(M,E)$ be the non–commutative conditional expectation given by equation (2.5). Observe that $\mathfrak{A}(M,E) \subseteq \mathfrak{D}(M,E)$ is an invariant subspace for γ_Φ and record the next formula derived from (4.3),

$$\gamma_\Phi(X) = X - 2\,\Sigma_\Phi\,\circ\,\Delta_\Phi\,\circ\,\mathfrak{D}(X), \quad X \in \mathfrak{A}(M,E).$$

Since $\mathfrak{A}_\Phi(M,E) = \mathfrak{D}_{\Phi,0}(M,E) \cap \mathfrak{A}(M,E)$, by part (iv) in Theorem D we get that π_Φ is the restriction of the spectral projection γ_Φ^+ to $\mathfrak{A}(M,E)$, hence

$$\pi_\Phi(X) = X - \Sigma_\Phi\,\circ\,\Delta_\Phi\,\circ\,\mathfrak{D}(X), \quad X \in \mathfrak{A}(M,E).$$

These remarks, Lemma 4.3, and Corollary 4.8 provide the next result.

Corollary 4.9. *Let* $\nabla_0 : \mathfrak{X}(M) \to \mathfrak{D}(M,E)$ *be a linear connection on the vector bundle* $\mathcal{E} = (E,\pi,M)$ *which preserves an* $\mathfrak{S}$*–structure* $\Phi \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{S},\mathfrak{A}(M,E))$. *The entire collection of linear connections on* $\mathcal{E}$ *that preserve structure* Φ *equals*

$$\{\,\nabla = \nabla_0 + \pi_\Phi\,\circ\,\Xi : \Xi \in \mathrm{Hom}_{C^\infty(M,\mathbb{R})}\,[\,\mathfrak{X}(M),\,\mathfrak{A}(M,E)\,]\,\}.$$

All previous results have counterparts for involutive $\mathfrak{S}$–structures on inner product vector bundles and explicit forms in terms of the elementary bases for $\mathfrak{S}$.

5 Related Themes, Comments, and References

An incipient use of ad hoc set up symbol and conjugation operators associated with geometric structures, such as product, complex, quaternionic, Clifford, metric, symplectic, and contact structures, originated in Martin [18, 19]. The sequels Martin [20, 21] point out additional incremental developments.

Structure manifolds of group with two–cocycle algebra Hom environments were introduced in Martin [24], as means of studying spaces of projective compact group representations in C^*–algebras. The simplest cases of cyclic groups of order two or higher yield Grassmann and flag manifolds of C^*–algebras. Such manifolds and their standard complex structures proved critical in generalizing the Cowen–Douglas theory, Cowen, Douglas [7], to a C^*–algebra setting. For details in this regard we refer to Martin [22, 23] and Martin, Salinas [41–43]. As another related research project, holomorphic mappings of several complex variables with values in Grassmann manifolds, Hermitian holomorphic vector bundles of finite or infinite rank, and Cowen–Douglas systems of Hilbert space operators, have been investigated in Martin [38] as the natural objects of interest in developing holomorphic spectral theory.

The general concepts of algebra environments, associated structure manifolds, Zariski tangent spaces to structure manifolds, derivations on algebra enviroments, and symbol and conjugation operators, were defined in Martin [39] and analyzed from algebraic geometry, topology, and differential geomety perspectives. An approach to the study of spaces of algebra homomorphisms as structure manifolds of Hom environments has been presented in Martin [40].

Clifford algebra Hom environments provide frameworks for addressing themes in Cliffford analysis and Spin geometry, two research areas concerned with the study of Dirac and Laplace operators in Euclidean and smooth Clifford vector bundle settings. The monographs by Anglés [2], Berline, Getzler, Vergne [3], Brackx, Delanghe, Sommen [4], Cnops [5], Colombo, Sabadini, Sommen, Struppa [6], Delanghe, Sommen, Souček [8], Gilbert, Murray [10], Gürlebeck, Sprössig [11], Lawson, Michelsohn [16], Louenesto [17], Rocha–Chavez, Shapiro, Sommen [49], are highly recommended sources of information in this regard. The chapters edited by Alpay, Colombo, Sabadini in the handbook Operator Theory [1], provide an excellent illustration of the full scope of past and current developments in quaternionic and Clifford analysis. Part of the history of the two areas is outlined in Martin [35].

For more applications of algebra environments related to Cliffford analysis, Spin geometry, harmonic analysis, and multivariable operator theory we direct the attention of our reader to the articles Martin [25–37] and Martin, Salinas [44]. The list of specific issues includes Cauchy–Pompeiu and Bochner–Martinelli–Koppelman integral representation formulas for operator–kernel couples with coefficients in a Banach algebra, maximal and fractional integral operators in Clifford analysis, generalizations of Ahlfors–Beurling and Alexander inequalities, quantitative Hartogs–Rosenthal theorems, Bochner–Weitzenböck and Bochner–Kodaira–Nakano identities, extensions of Putnam inequality and singular integral Riesz transforms models of seminormal systems of operators.

As a concluding comment, we want to mention the research projects concerned with the study of manifolds carrying special geometric structures, and refer to the articles by Gauduchon, Moroianu, Semmelmann [9], Hitchin [13], Moroianu, Semmelmann [45], Ornea, Piccinni [48], Vaisman [50], and Verbitsky [51]. Though our article does not address this theme, it highly motivated our inquiry.

References

[1] D. Alpay, F. Colombo, I. Sabadini, (editors), *Operator Theory*, Springer (2015).

[2] P. Anglés, *Conformal Groups in Geometry and Spin Structures*, Progress in Mathematical Physics **50**, Birkhäuser (2008).

[3] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Grundlehren der mathematischen Wissenschaften **298**, Springer (1992).

[4] F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis*, Pitman Research Notes in Mathematics Series **76**, Pitman (1982).

[5] J. Cnops, *An Introduction to Dirac Operators on Manifolds*, Progress in Mathematical Physics Series **24**, Birkhäuser (2002).

[6] F. Colombo, I. Sabadini, F. Sommen, D. C. Struppa, *Analysis of Dirac Systems and Computational Algebra*, Progress in Mathematical Physics Series **39**, Birkhäuser (2004).

[7] M. J. Cowen, R. G. Douglas, Complex geometry and operator theory, *Acta Math.* **141** (1978), 187–261.

[8] R. Delanghe, F. Sommen, V. Souček, *Clifford Algebra and Spinor Valued Functions*, Kluwer (1982).

[9] P. Gauduchon, A. Moroianu, U. Semmelmann, Almost complex structures on quaternion–Kähler manifolds and inner symmetric spaces, *Invent. Math.* **184** (2011), 389–403.

[10] J. E. Gilbert, M. A. M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge Studies in Advanced Mathematics **26**, Cambridge University Press (1991).

[11] K. Gürlebeck, W. Sprössig, *Quaternionic and Clifford Calculus for Physicists and Engineers*, Wiley (1997).

[12] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press (1978).

[13] N. Hitchin, Hyperkähler manifolds, *Séminaire Bourbaki* **34** (1991–1992), 137–166.

[14] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry* I, II, Willey–Interscience (1996).

[15] S. Lang, *Fundamentals of Differential Geometry*, Springer (1999).

[16] H. B. Lawson, M.-L. Michelsohn, *Spin Geometry*, Princeton Mathematical Series **38**, Princeton University Press (1989).

[17] P. Louenesto, *Clifford Algebras and Spinors*, Cambridge University Press (2001).

[18] M. Martin, Almost contact structures and derivations, *Rev. Roumaine Math. Pures Appl.* **26** (1981), 267–173.

[19] M. Martin, Geometric structures and derivations, *Rev. Roumaine Math. Pures Appl.* **29** (1984), 239–248.

[20] M. Martin, Geometric structures on vector bundles I. Geometric structures on vector spaces, *Stud. Cerc. Mat. / Math. Reports* **36** (1984), 113–142.

[21] M. Martin, Geometric structures on vector bundles II. Geometric structures and derivations, *Stud. Cerc. Mat. / Math. Reports* **36** (1984), 171–192.

[22] M. Martin, Hermitian geometry and involutive algebras, *Math. Z.* **188** (1985), 359–382.

[23] M. Martin, An operator theoretic approach to analytic functions into a Grassmann manifold, *Math. Balkanica N. S.* **1** (1987), 45–58.

[24] M. Martin, Projective representations of compact groups in C^*–algebras, in *Linear Operators in Function Spaces*, Operator Theory Advances and Applications **43**, Birkhäuser (1990), 237–253.

[25] M. Martin, Joint seminormality and Dirac operators, *Integral Equations Operator Theory* **30 (1)** (1998), 101–121.

[26] M. Martin, Higher–dimensional Ahlfors–Beurling inequalities in Clifford analysis, *Proc. Amer. Math. Soc.* **126** (1998), 2863–2871.

[27] M. Martin, Convolution and maximal operator inequalities, in *Clifford Algebras and their Applications in Mathematical Physics*, Progress in Mathematical Physics **19**, Birkhäuser (2000), 83–100.

[28] M. Martin, Self–commutator inequalities in higher dimension, *Proc. Amer. Math. Soc.* **130** (2002), 2971–2983.

[29] M. Martin, Spin geometry, Clifford analysis, and joint seminormality, in *Advances in Analysis and Geometry*, Trends in Mathematics Series **1**, Birkhäuser (2004), 227–255.

[30] M. Martin, Uniform approximation by solutions of elliptic equations and seminormality in higher dimension, in *Current Trends in Operator Theory and its Applications*, Operator Theory Advances and Applications **149**, Birkhäuser (2004), 387–406.

[31] M. Martin, Uniform approximation by closed forms in several complex variables, *Adv. Appl. Clifford Algebras* **19 (3–4)** (2009), 777–792.

[32] M. Martin, Deconstructing Dirac operators I. Quantitative Hartogs–Rosenthal theorems, in *More Progress in Analysis*, World Scientific (2009), 1065–1074.

[33] M. Martin, Deconstructing Dirac operators III. Dirac and semi–Dirac pairs, in *Topics in Operator Theory*, Operator Theory Advances and Applications **203**, Birkhäuser (2010), 347–362.

[34] M. Martin, Deconstructing Dirac operators II. Integral representation formulas, in *Hypercomplex Analysis and Applications*, Birkhäuser (2011), 195–211.

[35] M. Martin, Function spaces in quaternionic and Clifford analysis, in *Operator Theory*, Springer (2015), 1393–1422.

[36] M. Martin, Derivations and linear connections on Clifford vector bundles, *Adv. Appl. Clifford Algebras* **27** (2017), 2585–2605.

[37] M. Martin, Seminormal systems of operators in Clifford environments, *Opuscula Mathematica* **37 (1)** (2017), 81–107.

[38] M. Martin, Holomorphic spectral theory: A geometric approach, *Complex Analysis Operator Theory* **16** (2022), 1–39.

[39] M. Martin, Algebra environments I. Geometric and topological structures, *Rev. Roumaine Math. Pures Appl.* **69 (1)** (2024), 17–60.

[40] M. Martin, Algebra environments II. Algebra homomorphisms and derivations, *Linear Algebra and its Applications* **694 (1)** (2024), 30–51.

[41] M. Martin, N. Salinas, Differential geometry of generalized Grassmann manifolds in C^*-algebras, in *Operator Theory and Boundary Eigenvalue Problems*, Operator Theory Advances and Applications **80**, Birkhäuser (1995), 206–243.

[42] M. Martin, N. Salinas, Flag manifolds and the Cowen–Douglas theory, *J. Operator Theory* **38 (2)** (1997), 329–365.

[43] M. Martin, N. Salinas, The canonical complex structure of flag manifolds in a C^*-algebra, in *Non–Selfadjoint Operator Algebras, Operator Theory and Related Topics*, Operator Theory Advances and Applications **104**, Birkhäuser (1998), 173–187.

[44] M. Martin, N. Salinas, Weitzenböck type formulas and joint seminormality, in *Operator Theory for Complex and Hypercomplex Analysis*, Contemporary Mathematics **212** (1998), 157–167.

[45] A. Moroianu, U. Semmelmann, Clifford structures on Riemannian manifolds, *Adv. Math.* **228** (2011), 940–967.

[46] R. Narasimhan, *Analysis on Real and Complex Manifolds, 2nd Edition*, North-Holland Mathematical Library **35**, Elsevier Science Publishers (1991).

[47] L. Nicolaescu, *Lectures on the Geometry of Manifolds, 2nd Edition*, World Scientific (2007).

[48] L. Ornea, P. Piccinni, Locally conformally Kähler structures in quaternionic geometry, *Trans. Amer. Math. Soc.* **349 (2)** (1997), 641–655.

[49] R. Rocha–Chavez, M. Shapiro, F. Sommen, Integral Theorems for Functions and Differential Forms in $\mathbb{C}^m$, Research Notes in Mathematics **428**, Chapman & Hall (2002).

[50] I. Vaisman, Dirac structures and generalized complex structures on $\mathrm{T}M \times \mathbb{R}^h$, *Advances in Geom.* **7** (2007), 453–474.

[51] M. Verbitsky, Hypercomplex structures on Kähler manifolds, *Geometric and Functional Analysis* **15 (5)** (2005), 1275–1283.

Received: 19.10.2024
Accepted: 03.02.2025

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