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Closure operators for semitopogenous spaces
by
Tom Richmond[(1)], Josef Šlapal[(2)]

Abstract

Semitopogenous orders on a set X were introduced by Császár to provide a unified approach to topology, proximity, and uniformity. At least four closure operators associated with a semitopogenous order have been introduced by various authors. We provide a systematic comparison of these closure operators. Examples are presented to show their relative dependence and independence.

Key Words: Semitopogenous order, closure operator, topology, connected closure space.
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1 Introduction

Semitopogenous orders provide a single approach to study topologies, proximities, and uniformities. Along with stronger structures such as topogenous orders and collections of topogoneous orders, semitopogenous orders were introduced and systematically studied in Á. Császár's 1960 text *Fondements de la topologie générale*, which appeared in English [3] three years later. Semitopogenous orders have been studied since then, starting with investigations of basic properties [4, 12, 13]. Recent advances have included connections with category theory and closure operators [7, 8, 14].

Suppose $\sqsubset$ is a semitopogenous order on X. In a natural sense, $x \sqsubset U$ models the statement that U is a neighborhood of x. With this in mind, the definition of closure cl_C from [3] is based on the idea that $x \in cl_C(A)$ if and only if every neighborhood of x intersects A. In [11], the authors define a closure operator $cl_{\sqsubset}$ based on the idea that $x \in cl_{\sqsubset}(A)$ if and only if every neighborhood of x is a neighborhood of some point of A. In a topological space, the intersection of all open sets containing A is called the kernel of A, denoted $ker(A)$, and this is a closure operator. This idea is used to define a third closure operator $cl^{\sqsubset}$ from $\sqsubset$. In [4], Császár defines a topology $\mathcal{T}_{\sqsubset}$ on X using the idea that U is open if it is a neighborhood of each of its points. The Kuratowski closure operator $cl_{\mathcal{T}_{\sqsubset}}$ generated by this topology gives a fourth closure operator defined by $\sqsubset$. Our goal is to investigate and compare these four closure operators determined by a semitopogenous order $\sqsubset$ on a set X.

In Section 2 we formally define the closure operators and show some dependencies between them. In Section 3 we show how to generate a semitopogenous order $\sqsubset_p$ from a closure operator p on X and give special attention to the case where p is the Kuratowski closure operator associated with a given topology. In Section 4 we give several examples of topogenous orders which illustrate that the only dependencies between the four closure operators which always hold are those given in Section 2.

We start with some relevant definitions.

Given a nonempty set X, we will consider the following conditions which a binary relation $\sqsubset$ on the power set $\mathcal{P}(X)$ might satisfy:

(S1) $\emptyset \sqsubset \emptyset$, $X \sqsubset X$.

(S2) $A \sqsubset B$ implies $A \subseteq B$.

(S3) $A \subseteq A' \sqsubset B' \subseteq B$ implies $A \sqsubset B$.

(S4) $A \sqsubset B$ and $A' \sqsubset B'$ implies $A \cap A' \sqsubset B \cap B'$, and (S4-$\cap$)
$A \sqsubset B$ and $A' \sqsubset B'$ implies $A \cup A' \sqsubset B \cup B'$. (S4-$\cup$)

(S5) $A \sqsubset B$ implies there exists $C \subseteq X$ with $A \sqsubset C \sqsubset B$.

A relation $\sqsubset$ on $\mathcal{P}(X)$ satisfying (S1), (S2), and (S3) is a *semitopogenous order on* X, and a relation $\sqsubset$ on $\mathcal{P}(X)$ satisfying (S1), (S2), (S3), and (S4) is a *topogenous order on* X. A relation satisfying (S5) is said to be *interpolating.* If $\sqsubset$ is an interpolating topogenous order on X, Császár [3] calls $(X, \sqsubset)$ a *topogenous structure.* A *syntopogenous structure* on X is a collection $\mathcal{S} = \{\sqsubset_i\colon i \in I\}$ of topogenous orders on X, each satisfying (S1)–(S5). Following Császár [3], we may write $x \sqsubset B$ for $\{x\} \sqsubset B$.

A semitopogenous order $\sqsubset$ on X is *perfect* (*coperfect)* if $A_i \sqsubset B_i$ for all indices i in an arbitrary index set I implies $\bigcup_{i\in I} A_i \sqsubset \bigcup_{i\in I} B_i$ ($\bigcap_{i\in I} A_i \sqsubset \bigcap_{i\in I} B_i$). If $\sqsubset$ is perfect and coperfect, it is said to be *biperfect.*

One motivation for the introduction of topogenous structures was to generalize topologies. If τ is a topology on X, taking $A \sqsubset_\tau B$ if and only if $A \subseteq intB$ gives a perfect interpolating topogenous order $\sqsubset_\tau$.

Every topogenous order $\sqsubset$ on X has an associated *complementary topogenous order* $\sqsubset^c$ defined by $A \sqsubset^c B$ if and only if $X - B \sqsubset X - A$. If τ is a topology on X, $A \sqsubset^c_\tau B$ if and only if $clA \subseteq B$.

Given a set X, a function $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ will be called a *closure operator* if it satisfies these conditions:

grounded: $cl(\emptyset) = \emptyset$,
extensive: $A \subseteq cl(A)$ for all $A \subseteq X$,
monotone: $A \subseteq B \subseteq X \Rightarrow cl(A) \subseteq cl(B)$.

If cl is a closure operator on X, the pair (X, cl) is called a *closure space.* These closure operators were studied by E. Čech in his pioneering paper published (in Czech) as early as 1937 (see [5]). A closure operator cl is a *Kuratowski closure operator* if it also satisfies these conditions:

additive: $cl(A \cup B) = cl(A) \cup cl(B)$ for all $A, B \subseteq X$, and
idempotent: $cl(cl(A)) = cl(A)$ for all $A \subseteq X$.

Recall that Kuratowski closure operators cl on a set X are just those closure operators that are associated with topologies τ on X (given by families of open sets, i.e., $\tau \subseteq \mathcal{P}(X)$ where, for any $\mathcal{A} \subseteq \tau$, $\bigcup \mathcal{A} \in \tau$ and, for every finite $\mathcal{B} \subseteq \tau$, $\bigcap \mathcal{B} \in \tau$) such that $cl(A) = X - \bigcup\{B \in \tau;\ B \subseteq X - A\}$. Therefore, Kuratowski closure operators are often called *topological closures.*

The set $cl(A)$ is said to be the *closure* of A (under the closure operator cl). As usual, a set is said to be *closed* if it equals its closure. Given closure operators cl and cl' on X, we put $cl \leq cl'$ if $cl(A) \subseteq cl'(A)$ for every subset $A \subseteq X$ (cf. [5]). Evidently, $\leq$ is a partial order on the set of all closure operators on X. As usual, we write $cl < cl'$ if $cl \leq cl'$ but $cl \neq cl'$.

A *separation* of a topological space (X, τ) is a pair $\{A, B\}$ of nonempty subsets of X with $A \cup B = X$, $A \cap B = \emptyset$, and $A \cap cl(B) = B \cap cl(A) = \emptyset$. The topological space (X, τ) is *connected* if it has no separation. This definition applies to any set with a closure operator cl in which case we speak about a *cl-connected closure space*. Clearly X is connected if and only if there are no nonempty closed sets A, B with $A \cup B = X$, $A \cap B = \emptyset$.

For the topological concepts used we refer to [10].

2 The closure operators

From the motivating link between topologies and topogenous orders $A \sqsubset B$ if and only if $A \subseteq intB$, we may loosely think of $x \sqsubset U$ to mean U is a neighborhood of x. This suggests the following results, which are easily confirmed.

Theorem 2.1.

(a) If $\sqsubset$ is a semitopogenous order on X, then $cl_C : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$cl_C(A) = \{x : x \sqsubset U \Rightarrow U \cap A \neq \emptyset\}$$

is grounded, extensive, and monotone.

(b) If $\sqsubset$ is a semitopogenous order on X, then $cl_{\sqsubset} : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$cl_{\sqsubset}(A) = \{x \in X : x \sqsubset U \Rightarrow a \sqsubset U \quad \textit{for some } a \in A\}$$

is grounded, extensive, monotone, and idempotent.

(c) if $\sqsubset$ is a semitopogenous order on X, then $cl^{\sqsubset} : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$cl^{\sqsubset}(A) = \bigcap \{U \subseteq X : A \sqsubset U\}$$

is grounded, extensive, and monotone.

(d) If $\sqsubset$ satisfies (S1), (S3), and (S4-$\cap$), then

$$\mathcal{T}_{\sqsubset} = \{U \subseteq X : x \in U \Rightarrow x \sqsubset U\}$$

is a topology on X. The associated Kuratowski closure operator is denoted $cl_{\mathcal{T}_{\sqsubset}}$, and this is simply the topological closure associated with $\mathcal{T}_{\sqsubset}$.

The closure operator cl_C was introduced by Császár ([3], pp. 212–219), where he required $\sqsubset$ to satisfy (S1)–(S5). He used this definition in the setting of syntopogenous structures.

The closure operator $cl_{\sqsubset}$ was introduced by Richmond and Slapal [11], where a proof of Theorem 2.1(b) is given.

Versions of $cl^{\sqsubset}$ have been considered by Maki [9] (for $\sqsubset = \sqsubset_\tau$), Slapal, Richmond, and Iragi [14] (for topogenous orders on a poset, not necessarily $\mathcal{P}(X)$), and Holgate, Iragi, and Razafindrakoto [7] (for coperfect topogenous orders).

Császár [4] defined the topology $\mathcal{T}_{\sqsubset}$, assuming $\sqsubset$ satisfied (S1)–(S5).

The collection of $cl_{\sqsubset}$-closed (respectively, $cl^{\sqsubset}$-closed, cl_C-closed) sets in $(X, \sqsubset)$ will be denoted $\mathcal{F}_{\sqsubset}$ (respectively, $\mathcal{F}^{\sqsubset}$, $\mathcal{F}_C$). If $\mathcal{F}$ is a collection of subsets of X, $\mathcal{CF} = \{X - F : F \in \mathcal{F}\}$ is the collection of complements of elements of $\mathcal{F}$.

It is natural to ask whether the complements of the closed sets with respect to $cl_{\sqsubset}, cl_C$, or $cl^{\sqsubset}$ give the $\mathcal{T}_{\sqsubset}$-open sets. That is, we may ask whether $\mathcal{T}_{\sqsubset} = \mathcal{CF}_{\sqsubset}$ or $\mathcal{CF}_C$ or $\mathcal{CF}^{\sqsubset}$. This will not be true in general since the collection $\mathcal{CF}$ of complements of closed sets with respect to a closure operator cl is a topology if and only if cl is a Kuratowski closure operator. We have the following result relating the closure operators.

Theorem 2.2. *Suppose $\sqsubset$ is a semitopogenous order on X.*

(a) $cl_{\sqsubset} \leq cl_C \leq cl_{\mathcal{T}_{\sqsubset}}$.

(b) $\mathcal{T}_{\sqsubset} \subseteq \mathcal{CF}_C \subseteq \mathcal{CF}_{\sqsubset}$.

(c) $\mathcal{CT}_{\sqsubset} \subseteq \mathcal{F}_C \subseteq \mathcal{F}_{\sqsubset}$.

Proof. Let $A \subseteq X$. $cl_{\sqsubset}(A) \subseteq cl_C(A)$ is immediate from the definitions since $a \sqsubset U$ implies $a \in U$. For the corresponding inclusion of part (c), $\mathcal{F}_C \subseteq \mathcal{F}_{\sqsubset}$, note that $A \in \mathcal{F}_C \Rightarrow A = cl_C(A) \Rightarrow A \subseteq cl_{\sqsubset}(A) \subseteq cl_C(A) = A \Rightarrow A = cl_{\sqsubset}(A) \Rightarrow A \in \mathcal{F}_{\sqsubset}$. (c) and (b) are clearly equivalent, so we have shown the inclusions of (a), (b), and (c) relating to $cl_{\sqsubset}$ and cl_C.

We will now show $\mathcal{T}_{\sqsubset} \subseteq \mathcal{CF}_C$. Suppose $U \in \mathcal{T}_{\sqsubset}$. We want to show $X - U = cl_C(X - U)$. Suppose $y \in cl_C(X - U)$ and $y \in U$. Now $U \in \mathcal{T}_{\sqsubset}$ implies $y \sqsubset U$. By the definition of cl_C, $y \sqsubset U$ and $y \in cl_C(X - U)$ implies $U \cap (X - U) \neq \emptyset$, which is a contradiction. Thus, $y \in cl_C(X - U)$ implies $y \in X - U$ so $X - U$ is cl_C-closed. Thus, $\mathcal{T}_{\sqsubset} \subseteq \mathcal{CF}_C$, and it follows that $\mathcal{CT}_{\sqsubset} \subseteq \mathcal{F}_C$.

Now we will show that $cl_C(A) \subseteq cl_{\mathcal{T}_{\sqsubset}}(A)$. From $\mathcal{CT}_{\sqsubset} \subseteq \mathcal{F}_C$ and the fact that $cl_{\mathcal{T}_{\sqsubset}}$ is idempotent, we have $cl_{\mathcal{T}_{\sqsubset}}(A) \in \mathcal{CT}_{\sqsubset} \subseteq \mathcal{F}_C$. If cl_C is also idempotent, then $cl_C(A)$ would be the smallest cl_C-closed set containing A, so $cl_C(A) \subseteq cl_{\mathcal{T}_{\sqsubset}}(A)$. In general cl_C need not be idempotent, so $cl_C(A)$ need not be closed (see Example 4.2) and this argument may fail. However, in all cases, $A \subseteq cl_{\mathcal{T}_{\sqsubset}}(A) \in \mathcal{CT}_{\sqsubset} \subseteq \mathcal{F}_C$ so $cl_C(A) \subseteq cl_C(cl_{\mathcal{T}_{\sqsubset}}(A)) = cl_{\mathcal{T}_{\sqsubset}}(A)$, as needed. ∎

The examples of Section 4 will show that no other inclusions are realized between $cl_{\sqsubset}, cl_C, cl^{\sqsubset}$, and $cl_{\mathcal{T}_{\sqsubset}}$.

If τ is a topology on X, then $\mathcal{T}_{\sqsubset_\tau} = \tau$ and the closure operators cl_C, $cl_{\sqsubset_\tau}$, and $cl_{\mathcal{T}_{\sqsubset_\tau}}$ associated with $\sqsubset_\tau$ agree with the topological closure cl_τ. For $A \subseteq X$, the set $cl^{\sqsubset_\tau}(A) = \bigcap\{U : A \subseteq U, U \in \tau\}$ is called the *kernel* of A, denoted $ker(A)$. Sets which are equal to their kernel, that is, $cl^{\sqsubset_\tau}$-closed sets, are called Λ-sets in [9, 1, 6, 2]. If τ is T_1, then every set equals its kernel, so every set is $cl^{\sqsubset_\tau}$-closed, and in particular, $cl^{\sqsubset_\tau} \neq cl_\tau = cl_{\mathcal{T}_{\sqsubset_\tau}}$ if τ is not the discrete topology (i.e., the topology on X with all subsets of X open). If τ is regular (not necessarily T_1) and $x \notin cl_\tau(A)$, then there exists a τ-open neighborhood V of x disjoint from a τ-open set U containing A, so $x \notin cl^{\sqsubset_\tau}(A)$. This shows $cl^{\sqsubset_\tau}(A) \subseteq cl_\tau(A)$ if τ is regular.

Note that $cl_{\mathcal{T}_\sqsubset}$-connectedness (defined by the closure operator) is equivalent to $\mathcal{T}_\sqsubset$-connectedness (defined in the usual topological sense). We list an immediate corollary to Theorem 2.2.

Corollary 2.3. *Suppose $\sqsubset$ is a semitopogenous order on X. Then*

$$cl_\sqsubset\text{-connected} \Rightarrow cl_C\text{-connected} \Rightarrow cl_{\mathcal{T}_\sqsubset}\text{-connected.}$$

Proof. If $cl_1 \leq cl_2$, any cl_2 separation is a cl_1 separation. □

We will return to a discussion of implications between connectedness with respect to $cl_\sqsubset, cl_C, cl^\sqsubset$, and $cl_{\mathcal{T}_\sqsubset}$ at the end of Section 4.

Example 2.4. Theorem 2.1(d) gave sufficient conditions on $\sqsubset$ for $\mathcal{T}_\sqsubset = \{U \subseteq X : x \in U \Rightarrow x \sqsubset U\}$ to be a topology. These conditions are not necessary. On $X = \mathbb{R}$, define

$$A \sqsubset B \iff \begin{cases} |A| = 0 & \text{and } A \subset B \\ |A| = 1 & \text{and } A \subseteq B \\ |A| \geq 2 & \text{and } A \subseteq B, \sup B = \infty. \end{cases}$$

Clearly (S1) fails and (S2) holds. If $A \subseteq A' \sqsubset B' \subseteq B$, then $A \subseteq B$. If $|A| < 2$, then $A \sqsubset B$. If $|A| \geq 2$, then $|A'| \geq 2$ and $\sup B = \sup B' = \infty$, so $A \sqsubset B$. Thus, (S3) holds. (S4-$\cap$) fails: If $\mathbb{D}$ is the set of odd integers and $\mathbb{E}$ is the set of even integers, $A = \mathbb{E} \sqsubset \mathbb{E} = B$ and $A' = \{2,4\} \cup \mathbb{D} \sqsubset \{2,4\} \cup \mathbb{D} = B'$, but $A \cap A' = \{2,4\} \not\sqsubset \{2,4\} = B \cap B'$. (S4-$\cup$) also fails, since $\{1\} \sqsubset \{1\}$ and $\{2\} \sqsubset \{2\}$ but $\{1,2\} \not\sqsubset \{1,2\}$. (S5) holds: If $A \sqsubset B$ then $A \sqsubset A \sqsubset B$ if $|A| < 2$ and $A \sqsubset B \sqsubset B$ if $|A| \geq 2$.

Now $\mathcal{T}_\sqsubset = \{U \subseteq X : x \in U \Rightarrow x \sqsubset U\} = \mathcal{P}(\mathbb{R})$ is a topology even though (S1) and (S4-$\cap$) fail.

It is easy to confirm that for any $A \subseteq \mathbb{R}$, $cl_\sqsubset(A) = cl_C(A) = cl^\sqsubset(A) = cl_{\mathcal{T}_\sqsubset}(A) = A$.

3 A semitopogenous order $\sqsubset_p$ from a closure operator p

If τ is a topology on X, the Kuratowski closure operator $p = cl_\tau$ gives a well-known interpolating topogenous order $\sqsubset_p \equiv \sqsubset_\tau^c$ defined by $A \sqsubset_p B$ if and only if $p(A) \subseteq B$. However, p need not be a Kuratowski closure operator for $\sqsubset_p$ to be a semitopogenous order.

Theorem 3.1. *Suppose $p : \mathcal{P}(X) \to \mathcal{P}(X)$ is grounded, extensive, and monotone. For $A, B \subseteq X$, define $A \sqsubset_p B$ if and only if $p(A) \subseteq B$.*

(a) $\sqsubset_p$ is a coperfect semitopogenous order. In particular, $\mathcal{T}_{\sqsubset_p}$ is an Alexandroff topology (that is, a topology closed under formation of arbitrary intersections).

(b) $\sqsubset_p$ satisfies (S4-$\cup$) if and only if p is additive.

(c) $\sqsubset_p$ satisfies (S5) if and only if p is idempotent.

Proof. It is easy to verify that $\sqsubset_p$ satisfies (S1)–(S3) and thus is a semitopogenous order. Furthermore, since $\bigcap\{A_i : i \in I\} \subseteq A_{i_0}$ for any $i_0 \in I$, the monotonicity of p gives $p(\bigcap A_i : i \in I\}) \subseteq \bigcap\{p(A_i) : i \in I\}$, so

$$\begin{aligned} A_i \sqsubset_p B_i \quad \forall i \in I &\Rightarrow p(A_i) \subseteq B_i \quad \forall i \in I \\ &\Rightarrow p(\bigcap\{A_i : i \in I\}) \subseteq \bigcap\{p(A_i) : i \in I\} \subseteq \bigcap\{B_i : i \in I\} \\ &\Rightarrow \bigcap\{A_i : i \in I\} \sqsubset_p \bigcap\{B_i : i \in I\}. \end{aligned}$$

This shows that $\sqsubset_p$ is coperfect.

To see that $\mathcal{T}_{\sqsubset_p}$ is an Alexandroff topology, suppose $U_i \in \mathcal{T}_{\sqsubset_p}$ for all $i \in I$. Then $x \in \bigcap\{U_i : i \in I\} \Rightarrow x \in U_i \ \forall i \in I \Rightarrow x \sqsubset_p U_i \ \forall i \in I \Rightarrow x \sqsubset_p \bigcap\{U_i : i \in I\}$, so $\bigcap\{U_i : i \in I\} \in \mathcal{T}_{\sqsubset_p}$.

(b) and (c) are special cases of [14, Propositions 3–4]. □

Theorem 2.2(a) showed how some of the closure operators determined by a semitopogenous order $\sqsubset$ are ordered. If $\sqsubset_p$ is as above, in general, there are no additional order between $cl_{\sqsubset_p}$, cl_C, $cl^{\sqsubset}$, $cl_{\mathcal{T}_{\sqsubset_p}}$. All inclusions not mentioned in Theorem 2.2(a) fail for semitopogenous orders of form $\sqsubset_p$ for some p.

Theorem 3.2.

(a) If $p : \mathcal{P}(X) \to \mathcal{P}(X)$ is a grounded, extensive, monotone map, then $cl^{\sqsubset_p} = p$.

(b) If $\sqsubset$ is a semitopogenous order on X and $p = cl^{\sqsubset}$, then $\sqsubset \subseteq \sqsubset_p$ and $\sqsubset = \sqsubset_p$ if and only if $\sqsubset$ is coperfect.

Proof. (a) Let $A \subseteq X$. Then $A \sqsubset_p U \Rightarrow p(A) \subseteq U$, so $p(A) \subseteq \bigcap\{U : A \sqsubset_p U\} = cl^{\sqsubset_p}(A)$. Also, $A \sqsubset_p p(A) \Rightarrow p(A) \in \{U : A \sqsubset_p U\} \Rightarrow cl^{\sqsubset_p}(A) = \bigcap\{U : A \sqsubset_p U\} \subseteq p(A)$. Alternately, (a) follows from Theorem 1 and Proposition 2 of [14].

(b) If $\sqsubset$ satisfies (S1)–(S3), then $p = cl^{\sqsubset}$ is grounded, extensive, and monotone, so we may construct $\sqsubset_p = \sqsubset_{cl^{\sqsubset}}$. In general, we cannot expect that $\sqsubset = \sqsubset_{cl^{\sqsubset}}$, since we have seen $\sqsubset_p$ is coperfect for any grounded, extensive, monotone p. However, we have $A \sqsubset B \Rightarrow B \in \{U : A \sqsubset U\} \Rightarrow p(A) = cl^{\sqsubset}(A) = \bigcap\{U : A \sqsubset U\} \subseteq B \Rightarrow A \sqsubset_p B$, so $\sqsubset \subseteq \sqsubset_p$.

Coperfect is equivalent to $A \sqsubset B_i$ for all $i \in I$ implies $A \sqsubset \bigcap_{i\in I} B_i$, so if $\sqsubset$ is not coperfect, there exist sets $A, B_i (i \in I)$ with $A \sqsubset B_i$ for all i but $A \not\sqsubset \bigcap B_i$. Now $\{B_i\}_{i\in I} \subseteq \{U : A \sqsubset U\}$, so $p(A) = cl^{\sqsubset}(A) = \bigcap\{U : A \sqsubset U\} \subseteq \bigcap\{B_i\}_{i\in I}$. Now $A \sqsubset_p \bigcap\{B_i\}_{i\in I}$ but $A \not\sqsubset \bigcap\{B_i\}_{i\in I}$. Thus, if $\sqsubset$ is not coperfect, then $\sqsubset_p \not\subseteq \sqsubset$.

Suppose $\sqsubset$ is coperfect, $p = cl^{\sqsubset}$ and $A \sqsubset_p B$. Then $p(A) = cl^{\sqsubset}(A) = \bigcap\{U : A \sqsubset U\} \subseteq B$. Since $\sqsubset$ is coperfect, $A \sqsubset \bigcap\{U : A \sqsubset U\} \subseteq B$, and thus $A \sqsubset B$. This shows $\sqsubset_p \subseteq \sqsubset$, so $\sqsubset_p = \sqsubset$. □

If τ is a topology on X, then we may compare the topological closure $p = cl_\tau$ with the four additional closure operators $cl_{\sqsubset_p}, cl_C, cl^{\sqsubset_p}$, and $cl_{\mathcal{T}_{\sqsubset_p}}$ generated by $\sqsubset_p = \sqsubset_{cl_\tau}$. Before giving the relations between these closure operators in Theorem 3.4, we give a characterization of the collections of $\mathcal{T}_{\sqsubset_p}$-open and $\mathcal{T}_{\sqsubset_p}$-closed sets when $p = cl_\tau$. Each of these is realized as the collection of unions of closures of points, but the closures are taken with respect to different topologies.

Lemma 3.3. *(a) If $p : \mathcal{P}(X) \to \mathcal{P}(X)$ is a grounded, extensive, monotone map, then the collection of $\mathcal{T}_{\sqsubset_p}$-closed sets is $\mathcal{CT}_{\sqsubset_p} = \{\bigcup_{y\in Y} cl_{\mathcal{T}_{\sqsubset_p}}(y) : Y \subseteq X\}$.*

(b) If τ is a topology on X and $p = cl_\tau$, then the collection of $\mathcal{T}_{\sqsubset_p}$-open sets is $\mathcal{T}_{\sqsubset_p} = \{\bigcup_{y\in Y} cl_\tau(y) : Y \subseteq X\}$.

Proof. (a) By Theorem 3.1(a), $\mathcal{T}_{\sqsubset_p}$ is an Alexandroff topology, and this result holds for any Alexandroff topology. Specifically, since $\mathcal{T}_{\sqsubset_p}$ is an Alexandroff topology, $\bigcup_{a\in A} cl_{\mathcal{T}_{\sqsubset_p}}(a)$ is a closed set containing A, so $cl_{\mathcal{T}_{\sqsubset_p}}(A) \subseteq \bigcup_{a\in A} cl_{\mathcal{T}_{\sqsubset_p}}(a)$. But $a \in A \Rightarrow cl_{\mathcal{T}_{\sqsubset_p}}(a) \subseteq cl_{\mathcal{T}_{\sqsubset_p}}(A)$, so $\bigcup_{a\in A} cl_{\mathcal{T}_{\sqsubset_p}}(a) \subseteq cl_{\mathcal{T}_{\sqsubset_p}}(A)$.

(b) Suppose $U \in \mathcal{T}_{\sqsubset_p}$. Then $x \in U \Rightarrow x \sqsubset_p U \iff cl_\tau(x) \subseteq U$, so $\bigcup_{x\in U} cl_\tau(x) \subseteq U$. Clearly $U \subseteq \bigcup_{x\in U} cl_\tau(x)$, so $U = \bigcup_{x\in U} cl_\tau(x)$. This shows that $\mathcal{T}_{\sqsubset_p} \subseteq \{\bigcup_{y\in Y} cl_\tau(y) : Y \subseteq X\}$.

Suppose $U = \bigcup_{y\in Y} cl_\tau(y)$ for some $Y \subseteq X$. Now $x \in U \Rightarrow x \in cl_\tau(y)$ for some $y \in Y$ and thus $p(x) = cl_\tau(x) \subseteq cl_\tau(y) \subseteq U$. This shows $x \in U \Rightarrow x \sqsubset_p U$, so $U \in \mathcal{T}_{\sqsubset_p}$. □

Theorem 3.4. *Suppose τ is a topology on X and $p = cl_\tau$. Then*

$$cl^{\sqsubset_p} = cl_\tau \quad \textit{and } cl_{\sqsubset_p} = cl_C = cl_{\mathcal{T}_{\sqsubset_p}},$$

and in general, no other inclusions hold between $cl_\tau, cl^{\sqsubset_p}, cl_{\sqsubset_p}, cl_C$, and $cl_{\mathcal{T}_{\sqsubset_p}}$.

Proof. Let $A \subseteq X$. Then $cl^{\sqsubset_p}(A) = cl_\tau(A)$ follows from Proposition 3.2(a). Further,

$$\begin{aligned}
cl_{\sqsubset_p}(A) &= \{x : cl_\tau(x) \subseteq U \Rightarrow \exists a \in A, cl_\tau(a) \subseteq U\}\\
&= \{x : \exists a \in A, cl_\tau(a) \subseteq cl_\tau(x)\} \text{ (since } cl_\tau(x) \subseteq U = cl_\tau(x))\\
&= \{x : cl_\tau(x) \cap A \neq \emptyset\} \text{ (since } a \in cl_\tau(x) \iff cl_\tau(a) \subseteq cl_\tau(x)\ )\\
&= cl_C(A).
\end{aligned}$$

If $x \in cl_C(A)$, then $cl_\tau(X) \cap A \neq \emptyset$. If $V \in \mathcal{T}_{\sqsubset_p}$ is a neighborhood of x, then from the definition of $\mathcal{T}_{\sqsubset_p}$, $cl_\tau(x) \subseteq V$, and it follows that $V \cap A \neq \emptyset$. Since every $\mathcal{T}_{\sqsubset_p}$-neighborhood of x intersects A, we have $x \in cl_{\mathcal{T}_{\sqsubset_p}}(A)$. Thus, $cl_C(A) \subseteq cl_{\mathcal{T}_{\sqsubset_p}}(A)$.

To see $cl_{\mathcal{T}_{\sqsubset_p}}(A) \subseteq cl_C(A)$, suppose $x \in cl_{\mathcal{T}_{\sqsubset_p}}(A)$. By Lemma 3.3(a), $cl_{\mathcal{T}_{\sqsubset_p}}(A) = \bigcup_{a\in A} cl_{\mathcal{T}_{\sqsubset_p}}(a)$, so $x \in cl_{\mathcal{T}_{\sqsubset_p}}(a)$ for some $a \in A$. Thus, every $\mathcal{T}_{\sqsubset_p}$-open neighborhood of x contains a. By Lemma 3.3(b), $cl_\tau(x)$ is a $\mathcal{T}_{\sqsubset_p}$-open neighborhood of x, so $a \in cl_\tau(x)$. Since $cl_C(A) = \{x : cl_\tau(x) \cap A \neq \emptyset\}$, we have $x \in cl_C(A)$, as needed.

Example 4.1 below shows that no other inclusions hold. □

Theorem 3.4 shows that every topology τ equals $\mathcal{CF}^{\sqsubset_p}$ for $p = cl_\tau$. Theorem 3.1(a) shows $\mathcal{T}_{\sqsubset_{cl_\tau}}$ is Alexandroff, so $\mathcal{T}_{\sqsubset_{cl_\tau}} \neq \tau = \mathcal{CF}^{\sqsubset_{cl_\tau}}$ if τ is not Alexandroff.

While $cl_{\mathcal{T}_{\sqsubset_p}} = cl_C$ if $p = cl_\tau$ for some topology τ on X, the result does not hold if p is only grounded, extensive, and monotone. See Example 4.2.

We note that if $p : \mathcal{P}(X) \to \mathcal{P}(X)$ is grounded, extensive, and monotone, defining $A \sqsubset_p^* B$ if and only if $A = B = \emptyset$ or $p(A) \subset B$ (that is, $p(A) \subseteq B$ and $p(A) \neq B$) gives a semitopogenous order.

4 Examples

In this section, we present several examples to show the variety of situations which can be modeled by semitopogenous orders, and to show that the order relations between the closure operators shown in the previous sections are the only ones which always hold.

The comments following Theorem 2.2 show that if τ is the discrete topology on a nonempty set X and $\sqsubset_\tau$ is the associated topogenous order (namely, $\sqsubset_\tau = \subseteq$), then $cl_C(A) = cl_{\sqsubset_\tau}(A) = cl^{\sqsubset_\tau}(A) = cl_{\mathcal{T}_{\sqsubset_\tau}}(A) = A$. Thus, equality between any pair of the closure operators is possible. Example 4.6 below shows that all four of the closure operators may be distinct.

By Theorem 2.2, we always have $cl_C \leq cl_{\mathcal{T}_\sqsubset}, cl_\sqsubset \leq cl_C$, and $cl_\sqsubset \leq cl_{\mathcal{T}_\sqsubset}$. The examples below show that the orders $\leq$ prescribed by Theorem 2.2 may be strict, and when comparing distinct pairs of closure operators not prescribed by Theorem 2.2, any result $(<, =, ||, >)$ regarding the partial order $\leq$ may occur. (As usual, we use the symbol $||$ to indicate that a pair of closure operators is incomparable, i.e., $cl||cl'$ means that $cl \not\leq cl'$ and $cl' \not\leq cl$.)

Example 4.1. Consider the topology $\tau = \{\emptyset, \{0\}, \{0,1\}\}$ on the set $X = \{0,1\}$. Letting $p = cl_\tau$ and noting that $\mathcal{T}_{\sqsubset_p} = \{\emptyset, \{1\}, \{0,1\}\}$, and $cl_C(a) = \{x : a \in cl_\tau(x)\}$, we obtain the closures of points shown in Table 1.

$cl_\tau(0) = cl^{\sqsubset_p}(0) = X$	$cl_\tau(1) = cl^{\sqsubset_p}(1) = \{1\}$
$cl_{\mathcal{T}_{\sqsubset_p}}(0) = \{0\}$	$cl_{\mathcal{T}_{\sqsubset_p}}(1) = X$
$cl_C(0) = \{0\}$	$cl_C(1) = X$

Table 1: Closures for $\sqsubset_p$ where $p = cl_\tau$ for $\tau = \{\emptyset, \{0\}, \{0,1\}\}$ on $X = \{0,1\}$.

Together with the results of Theorem 3.4, this shows that $cl^{\sqsubset_p}(0) = cl_\tau(0) \not\subseteq cl_C(0) = cl_{\mathcal{T}_{\sqsubset_p}}(0) = cl_{\sqsubset_p}(0)$ and $cl_C(1) = cl_{\mathcal{T}_{\sqsubset_p}}(1) = cl_{\sqsubset_p}(1) \not\subseteq cl_\tau(1) = cl^{\sqsubset_p}(1)$.

In summary, $cl^{\sqsubset_p} \,||\, cl_{\mathcal{T}_{\sqsubset_p}} = cl_C = cl_{\sqsubset_p}$.

Example 4.2. For a fixed $\varepsilon > 0$ and any $A \subseteq \mathbb{R}$, let A_ε be the ε-fattening of A, i.e., $A_\varepsilon = \bigcup\{(a-\varepsilon, a+\varepsilon) : a \in A\}$. Define $A \sqsubset_\varepsilon B \iff A_\varepsilon \subseteq B$. Note that $\sqsubset_\varepsilon$ has the form $\sqsubset_p$ where $p(A) = A_\varepsilon$.

$\sqsubset_\varepsilon$ satisfies (S1)–(S4) but (S5) fails (consider $\{0\} \sqsubset_\varepsilon (-\varepsilon, \varepsilon)$).

For $A \subseteq \mathbb{R}$, $cl_{\sqsubset_\varepsilon}(A) = A$, so $\mathcal{F}_{\sqsubset_\varepsilon} = \mathcal{P}(\mathbb{R})$. To show this, suppose $A \subseteq \mathbb{R}$ and $x \in cl_{\sqsubset_\varepsilon}(A)$. Now $x \sqsubset_\varepsilon (x-\varepsilon, x+\varepsilon)$, so there exists $a \in A$ with $a \sqsubset_\varepsilon (x-\varepsilon, x+\varepsilon)$, so $(a-\varepsilon, a+\varepsilon) \subseteq (x-\varepsilon, x+\varepsilon)$ and thus $x = a \in A$. Hence $cl_{\sqsubset_\varepsilon}(A) \subseteq A$, so equality follows since $cl_{\sqsubset_\varepsilon}$ is extensive.

For $A \subseteq \mathbb{R}$, $cl_C(A) = A_\varepsilon = cl^{\sqsubset_\varepsilon}(A)$, so $\mathcal{F}_C = \{\emptyset, \mathbb{R}\} = \mathcal{F}^{\sqsubset}$. Since $\sqsubset_\varepsilon$ has form $\sqsubset_p$ for the closure operator $p(A) = A_\varepsilon$, $cl^{\sqsubset_\varepsilon} = p(A) = A_\varepsilon$ by Theorem 3.2(a). Indeed,

$$\begin{aligned} cl_C(A) &= \{x : (x-\varepsilon, x+\varepsilon) \subseteq U \Rightarrow \exists a \in A \cap U\} \\ &= \{x : (x-\varepsilon, x+\varepsilon) \cap A \neq \emptyset\} \\ &= \{x : \exists a \in A, x \in (a-\varepsilon, a+\varepsilon)\} \\ &= \bigcup_{a \in A} (a-\varepsilon, a+\varepsilon) \\ &= A_\varepsilon. \end{aligned}$$

$\mathcal{T}_{\sqsubset_\varepsilon} = \{\emptyset, \mathbb{R}\}$.

Also note that if $a \sqsubset_\varepsilon B$ for all $a \in A$, then $A \sqsubset_\varepsilon B$, so $\sqsubset_\varepsilon$ is perfect. For this semitopogenous order, we have $cl_{\sqsubset_\varepsilon} < cl_c = cl^{\sqsubset_\varepsilon} < cl_{\mathcal{T}_{\sqsubset_\varepsilon}}$.

Example 4.3. On a set X, if S is a nonempty subset of X, define $A \sqsubset_S B$ if and only if $A = \emptyset$ or $A \cup S \subseteq B$. It is easily verified that $\sqsubset_S$ satisfies (S1)-(S5).

We have $\mathcal{T}_{\sqsubset_S} = \{U \subseteq X : S \subseteq U\} \cup \{\emptyset\}$, which is the Alexandroff topology denoted by *Super*(S) in [10].

If $A \neq \emptyset$, then $cl^{\sqsubset_S}(A) = A \cup S = p(A)$, so $\mathcal{F}^{\sqsubset_S} = \mathcal{T}_{\sqsubset_S}$ and $\mathcal{C}\mathcal{F}^{\sqsubset_S} = \mathcal{C}\mathcal{T}_{\sqsubset_S} = \{X\} \cup \{U \subseteq X : U \cap S = \emptyset\}$ is the Alexandroff topology called *Disjoint*(S) in [10].

If $A \cap S = \emptyset$ and $x \in cl_{\sqsubset_S}(A)$, then $x \sqsubset_S U = \{x\} \cup S$ so there exists $a \in A$ with $\{a\} \cup S \subseteq \{x\} \cup S$. Since $a \notin S$, we have $x = a$, so $cl_{\sqsubset_S}(A) = A$. If there exists $a_0 \in A \cap S$ then for any $x \in X$, $x \sqsubset_S U \Rightarrow \{x\} \cup S \subseteq U \Rightarrow S \subseteq U \Rightarrow \{a_0\} \cup S \subseteq U \Rightarrow a_0 \sqsubset_S U$, so $cl_{\sqsubset_S}(A) = X$. A similar argument gives $cl_C(A)$, and considering sets of $\mathcal{C}\mathcal{T}_{\sqsubset_S}$ =*Disjoint*(S), we have

$$cl_C(A) = cl_{\sqsubset_S}(A) = cl_{\mathcal{T}_{\sqsubset_S}}(A) = \begin{cases} A & \text{if } A \cap S = \emptyset \\ A \cup S & \text{if } A \cap S \neq \emptyset. \end{cases}$$

In particular, $\mathcal{T}_{\sqsubset_S} = \mathcal{F}^{\sqsubset_S} = \mathcal{C}\mathcal{F}_C = \mathcal{C}\mathcal{F}_{\sqsubset_S} = Super(S)$, or $\mathcal{F}_C = \mathcal{F}_{\sqsubset_S} = \mathcal{C}\mathcal{T}_{\sqsubset_S} = \mathcal{C}\mathcal{F}^{\sqsubset_S} =$ *Disjoint*(S).

With $X = \mathbb{R}$ and $S = [0,1]$, we see that $cl^{\sqsubset_S}([2,3]) = p([2,3]) = [0,1] \cup [2,3] \not\subseteq [2,3] = cl_C([2,3]) = cl_{\sqsubset_S}([2,3]) = cl_{\mathcal{T}_{\sqsubset_S}}([2,3])$. However, if $A \cap S \neq \emptyset$, then all four closures equal $p(A) = A \cup S$.

Since there are no disjoint nonempty sets in $\mathcal{T}_{\sqsubset_S} = \mathcal{F}^{\sqsubset_S}$ or in $\mathcal{F}_C = \mathcal{F}_{\sqsubset_S}$, each of these is connected.

For this example, we have $cl_C = cl_{\sqsubset_S} = cl_{\mathcal{T}_{\sqsubset_S}} < cl^{\sqsubset_S}$.

Example 4.4. On $\mathbb{R}$, define $A \sqsubset B$ if and only if $A = \emptyset$ or there is $b \in \mathbb{R}$ such that $A \cup (b, \infty) \subseteq B$. It is easy to check that $\sqsubset$ satisfies (S1)–(S5) and is perfect. It is not biperfect: $\mathbb{N} \sqsubset \mathbb{N} \cup (n, \infty)$ for all $n \in \mathbb{N}$ but $\mathbb{N} \not\sqsubset \mathbb{N} = \bigcap_{n \in \mathbb{N}} \mathbb{N} \cup (n, \infty)$. This topogenous order $\sqsubset$ is not of form $\sqsubset_p$ for any closure operator p.

$\mathcal{T}_{\sqsubset} = \{U \subseteq \mathbb{R} : \exists b \in \mathbb{R}, (b, \infty) \subseteq U\} \cup \{\emptyset\}$. Thus the complements of $\mathcal{T}_{\sqsubset}$-open sets are $\mathbb{R}, \emptyset$, and the sets which are bounded above and we have

$$cl_{\mathcal{T}_{\sqsubset}}(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \mathbb{R} & \text{if } \sup A = \infty \\ A & \text{if } A \text{ is bounded above.} \end{cases}$$

We will show $cl_{\sqsubset} = cl_{\mathcal{T}_{\sqsubset}}$. From the definition of $cl_{\sqsubset}$ we have $cl_{\sqsubset}\emptyset = \emptyset$. Suppose $\sup A = \infty$ and $x \in \mathbb{R}$. Now $x \sqsubset U \Rightarrow \exists b$ s.t. $\{x\} \cup (b, \infty) \subseteq U$. For $a \in A \cap (b, \infty)$, we have $a \sqsubset U$, so $x \in cl_{\sqsubset}(A)$, so $cl_{\sqsubset}(A) = \mathbb{R}$. Now suppose A is bounded above. To see $cl_{\sqsubset}(A) = A$, it suffices to show that $x \notin A \Rightarrow x \notin cl_{\sqsubset}(A)$. Suppose $x \notin A$. Then $x \sqsubset \{x\} \cup (\sup A, \infty)$. If $x \in cl_{\sqsubset}(A)$, then there would be an $a \in A$ with $a \sqsubset \{x\} \cup (\sup A, \infty)$, which gives the contradiction that $a \in \{x\} \cup (\sup A, \infty) \subseteq \mathbb{R} - A$.

Slight modifications of the paragraph above (or Theorem 2.2) show that $cl_C(A) = cl_{\sqsubset}(A)$. Thus, $\mathcal{F}_{\sqsubset} = \mathcal{F}_C = \mathcal{CT}_{\sqsubset} = \{\emptyset, \mathbb{R}\} \cup \{A \subseteq \mathbb{R} : A \text{ is bounded above}\}$.

We will now show that $cl^{\sqsubset}(A) = A$ for all $A \subseteq \mathbb{R}$. Suppose A is bounded above. For $n \in \mathbb{N}$, let $b_n = \max A + n$. Now $A \sqsubset A \cup (b_n, \infty)$, so $cl^{\sqsubset}(A) \subseteq \bigcap_{n\in\mathbb{N}}(A \cup (b_n, \infty)) = A$. Suppose A is unbounded above and a_n is a strictly increasing sequence in A diverging to ∞. Now $cl^{\sqsubset}(A) \subseteq \bigcap_{n\in\mathbb{N}}(A \cup (a_n, \infty)) = A$. Thus, $\mathcal{F}^{\sqsubset} = \mathcal{P}(\mathbb{R})$.

If $\{A, B\}$ is a partition of $\mathbb{R}$, one of the sets must be unbounded above and its $\mathcal{T}_{\sqsubset}$-, $cl_{\sqsubset}$-, or cl_C- closure will not be disjoint from the the other set, so $\mathbb{R}$ is connected in these settings. $\mathbb{R}$ is not $cl^{\sqsubset}$-connected.

Here we have $cl^{\sqsubset} < cl_C = cl_{\sqsubset} = cl_{\mathcal{T}_{\sqsubset}}$.

Example 4.5. (See [11].) On $X = \mathbb{R}$, define $A \sqsubset B$ if and only if $A \subseteq B$ and A is finite or $B = \mathbb{R}$. It is easy to see that $\sqsubset$ satisfies (S1)–(S5). However, $\sqsubset$ is not perfect: $\{x\} \sqsubset \mathbb{R}$ for all $x \in [0, 1]$, but $[0, 1] = \bigcup_{x\in[0,1]}\{x\} \not\sqsubset \mathbb{R}$. It is routine to verify that $\mathcal{F}_{\sqsubset} = \mathcal{F}_C = \mathcal{T}_{\sqsubset} = \mathcal{CT}_{\sqsubset} = \mathcal{P}(\mathbb{R})$.

If A is finite, $A \sqsubset A$ so $cl^{\sqsubset}(A) = A$. However, if A is infinite, $A \sqsubset U$ if and only if $U = \mathbb{R}$, so $cl^{\sqsubset}(A) = \mathbb{R}$. Thus, $cl^{\sqsubset}$ gives the closed sets of the cofinite topology.

Any partition $\{A, B\}$ of $\mathbb{R}$ is a separation in $\mathcal{T}_{\sqsubset}$, $cl_{\sqsubset}$, and cl_C. But since one of A or B must be infinite, one must have closure equal to $\mathbb{R}$, which is not disjoint from the other. Thus, $\mathbb{R}$ is $cl^{\sqsubset}$-connected.

If τ is a topology on X, then $U \in \mathcal{T}_{\sqsubset_\tau} = \tau$ if and only if $U \sqsubset U$. However, in this example the set $U = (0, 1)$ is $\mathcal{T}_{\sqsubset}$-open but $U \not\sqsubset U$.

We have $cl_C = cl_{\sqsubset} = cl_{\mathcal{T}_{\sqsubset}} < cl^{\sqsubset}$.

Our final example gives a semitopogenous order in which all four of the closure operators are distinct.

Example 4.6. On $\mathcal{P}(\mathbb{N})$, define $A \sqsubset B$ if and only if $A = \emptyset$ or $[\min A, 2\sup\ A] \cap \mathbb{N} \subseteq B$. For convenience, we will interpret all intervals in this example—including those of form $[a, \infty]$—to be subsets of $\mathbb{N}$ and omit the "$\cap\mathbb{N}$". This semitopogenous order arises as $\sqsubset_p$ for the function $p(A) = [\min A, 2\sup A]$ for nonempty sets A and $p(\emptyset) = \emptyset$, which is grounded, monotone, and extensive, but not additive nor idempotent.

By Theorem 3.1, $\sqsubset$ satisfies (S1)–(S3) and it is easy to see that it satisfies (S4-$\cap$). The example of $A = \{2\} \sqsubset \{2, 3, 4\} = B$, $A' = \{8\} \sqsubset \{8, 9, \ldots, 16\} = B'$ shows $A \cup A' \not\sqsubset B \cup B'$ (since $\{2, \ldots, 5, 6, 7, \ldots, 16\} \not\subseteq B \cup B'$), so (S4-$\cup$) is not satisfied. (S5) also fails: $A = \{2\} \sqsubset \{2, 3, 4\} = B$, and $A \sqsubset C \Rightarrow B \subseteq C$, and thus $C \sqsubset D \Rightarrow \{2, 3, 4, 5, 6, 7, 8\} \subseteq D$, so $A \sqsubset C \sqsubset B$ is not possible.

$cl_{\sqsubset}(A) = \{n : n \sqsubset U \Rightarrow a \sqsubset U \text{ for some } a \in A\} = \{n : [n, \ldots, 2n] \subseteq U \Rightarrow [a, \ldots, 2a] \subseteq U \text{ for some } a \in A\} = \{n : n = a \in A\} = A$, so $\mathcal{F}_{\sqsubset} = \mathcal{P}(A)$.

$cl_C(A) = \{n : [n, \ldots, 2n] \subseteq U \Rightarrow U \cap A \neq \emptyset\} = \{n : [n, \ldots, 2n] \cap A \neq \emptyset\} = \bigcup_{a\in A}[\lceil \frac{a}{2} \rceil, a]$. Thus, $A = cl_C(A)$ if and only if $A = \emptyset$, or $A = \mathbb{N}$, or $A = [1, \ldots, m]$ for some $m \in \mathbb{N}$, so $\mathcal{F}_C$ is the left ray topology on $\mathbb{N}$.

$cl^{\sqsubset}(A) = p(A)$ and it follows that $\mathcal{F}^{\sqsubset} = \{[a, \infty) : a \in \mathbb{N}\} \cup \{\emptyset\}$ is the right ray topology on $\mathbb{N}$.

$\mathcal{T}_{\sqsubset} = \{U : n \in U \Rightarrow [n, \ldots, 2n] \subseteq U\} = \{[a, \infty) : a \in \mathbb{N}\} \cup \{\emptyset\}$, the right ray topology. Thus, $cl_{\mathcal{T}_{\sqsubset}}(A) = [1, \sup A]$ and $\mathcal{C}\mathcal{T}_{\sqsubset} = \mathcal{F}_C$. Even though $\mathcal{C}\mathcal{T}_{\sqsubset} = \mathcal{F}_C$, it is not the case that $cl_{\mathcal{T}_{\sqsubset}}(A) = cl_C(A)$, since $cl_C(A)$ need not be in $\mathcal{F}_C$.

For example, with $A = \{7, 20\}$, $cl_{\sqsubset}(A) = A$, $cl_{\mathcal{T}_{\sqsubset}}(A) = [1, 20]$, $cl_C(A) = [4, 7] \cup [10, 20]$, and $cl^{\sqsubset}(A) = [7, 40]$, so all four closures are distinct.

One may check that $cl_{\sqsubset}$ is not connected, but the other three closure operators are connected.

In this example, $cl_{\sqsubset} < cl_C < cl_{\mathcal{T}_{\sqsubset}}$ and $cl_{\sqsubset} < cl^{\sqsubset} \,||\, cl_C, cl_{\mathcal{T}_{\sqsubset}}$.

Connectedness relations are summarized in the diagrams below.

$$cl_{\sqsubset}\text{-connected} \; \overset{\text{Cor. 2.3}}{\Longrightarrow} \; \underset{\text{Ex. 4.4}}{\not\Longleftarrow} \; cl_C\text{-connected} \; \overset{\text{Cor. 2.3}}{\Longrightarrow} \; cl_{\mathcal{T}_{\sqsubset}}\text{-connected} \; \overset{\text{Ex. 4.4}}{\not\Longrightarrow} \; \underset{\text{Ex. 4.5}}{\not\Longleftarrow} \; cl^{\sqsubset}\text{-connected}$$

$$cl_{\sqsubset}\text{-connected} \; \overset{\text{Ex. 4.4}}{\not\Longrightarrow} \; \underset{\text{Ex. 4.5}}{\not\Longleftarrow} \; cl^{\sqsubset}\text{-connected}$$

It is an open problem whether $cl_{\mathcal{T}_{\sqsubset}}$-connected implies cl_C-connected.

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(1) Department of Mathematics, Western Kentucky University,
1906 College Heights Blvd., Bowling Green, KY 42101 USA
E-mail: tom.richmond@wku.edu

(2) Institute of Mathematics, Brno University of Technology,
Technická 2, 616 69 Brno, Czech Republic
E-mail: slapal@fme.vutbr.cz