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Lifting multiplicative lattices to ideal systems

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Dedicated to late Professors Daniel D. Anderson and Franz Halter-Koch

Abstract

We present a mechanism which lifts a multiplicative lattice to a (weak) ideal system on some monoid.

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1 Introduction

A *multiplicative lattice* is a complete lattice with least element 0 and greatest element 1, on which there is defined a commutative completely join distributive monoid operation whose identity is 1. We write simply *lattice* to mean a multiplicative lattice. By a *monoid* we mean a commutative monoid with identity element 1 and zero element 0.

A (weak) ideal system on some monoid (see Definition 2.1) gives the multiplicative lattice of its r-ideals (see Theorem 2.2). In this short paper we take the inverse direction providing a lifting procedure of multiplicative lattices to (weak) ideal systems. This procedure is inspired by the work of Aubert [3] and Lediaev [6] where there are results on lifting multiplicative lattices to x-systems.

We obtain the following results. Let L be a lattice and H a submonoid of L generating L as a lattice (such H is named in this paper a *wire*, see Definition 2.3). Then H gives a weak ideal system r on H (Theorem 2.4, Corollary 2.5 and Proposition 2.7). This r is an ideal system iff H is a so-called *M-wire* (see Definition 2.3). A lattice which is liftable to an ideal system is generated by meet principal elements, while a lattice domain which is generated by principal elements is liftable to an ideal system (Proposition 2.7). See the definition for "(meet) principal element" in the next paragraph. In Proposition 2.9 we investigate some M-wires of the lattice $\mathbb{N}$ (with usual number multiplication where $\bigvee = gcd$ and $\bigwedge = lcm$) given by the norm function of a ring of quadratic integers. As an application of our results, we give a natural procedure to associate to a given lattice L another lattice L' generated by compact elements (see Remark 2.11 and Example 2.12).

Let L be a lattice. Denote by $\vee$ resp. $\wedge$ its join resp. meet. If $a, b \in L$, we denote by $[a, b]$ the interval $\{x \in L | a \leq x \leq b\}$. For $a, b \in L$, $(a : b)$ is the join of all $y \in L$ with $by \leq a$. Recall the following definitions due to Dilworth [4]. An element $x \in L$ is said to be *meet principal* if $a \wedge xb = x((a : x) \wedge b)$ for all $a, b \in L$. Next x is called *weak meet principal* if the preceding equality holds for all $a \in L$ and $b = 1$. An element $x \in L$ is said to be *join principal* if $a \vee (b : x) = (ax \wedge b) : x$ for all $a, b \in L$. Next x is called *weak join principal* if the preceding equality holds for all $a \in L$ and $b = 0$. Finally $x \in L$ is called *(weak) principal* if it is both (weak) meet principal and (weak) join principal.

An element $x \in L$ is said to be *compact* if $x \leq \bigvee A$ with $A \subseteq L$ implies $x \leq \bigvee B$ for some finite subset B of A. We say that a subset C of L *generates* L if every element of L is a join of some elements in C. Any undefined notation or terminology is standard as in [5] or [1].

2 Results

We recall the definition of a (weak) ideal system cf. [5, Chapter 2].

Definition 2.1. *Let H be a monoid. A* weak ideal system *on H is a map $r : \mathcal{P}(H) \to \mathcal{P}(H)$ satisfying the following axioms:*

$(s1)$ $XH \subseteq X_r$ *for all* $X \subseteq H$,
$(s2)$ $X \subseteq Y \subseteq H$ *implies* $X_r \subseteq Y_r$
$(s3)$ $(X_r)_r = X_r$ *for all* $X \subseteq H$,
$(s4)$ $cX_r \subseteq (cX)_r$ *for all* $X \subseteq H$ *and* $c \in H$.

A weak ideal system r is called an ideal system *if equality always holds in $(s4)$. Also a (weak) ideal system r is said to be* finitary *if*

$(s5)$ $X_r = \bigcup \{Z_r | Z$ *finite subset of* $X\}$ *for all* $X \subseteq H$.

The elements in the image of r are called r-ideals.

The next result follows immediately from [5, Propositions 2.1 and 2.3] and definitions.

Theorem 2.2. *Let H be a monoid and r a weak ideal system on H. Then the set*

$$I_r(H) := \{X_r \mid X \subseteq H\}$$

of all r-ideals of H is a lattice with respect to the following operations

multiplication: $(X, Y) \mapsto (XY)_r$ *for all* $X, Y \in I_r(H)$,
join: $\bigvee \Gamma := (\bigcup \Gamma)_r$ *for all* $\Gamma \subseteq I_r(H)$,
meet: $\bigwedge \Gamma := \bigcap \Gamma$ *for all* $\Gamma \subseteq I_r(H)$,

where $\bigcup \Gamma$ resp. $\bigcap \Gamma$ are the union resp. intersection of all members of Γ.
If r is finitary, then $S =: \{\{a\}_r \mid a \in H\}$ is a generating submonoid of the lattice $I_r(H)$ consisting of compact elements.

Let L be a lattice. We look for a weak ideal system r whose r-ideal lattice $I_r(H)$ is isomorphic to L. In this case we say that r is a *lifting* of L or that L is *liftable* (to r). Getting inpiration from [3] and [6], we introduce the following concept.

Definition 2.3. *Let L be a lattice. By a* wire *$H \subseteq L$ we mean a submonoid of L which generates L as lattice. A wire H is called an* M-wire *if it satisfies the following condition:*

(M) *if $s \leq ta$ with $s, t \in H$ and $a \in L$, then $s = tu$ for some $u \in H \cap [0, a]$.*

The next theorem is the main result of the paper.

Theorem 2.4. *Let H be a wire of a lattice L. Then the map*

$$r : \mathcal{P}(H) \to \mathcal{P}(H) \text{ given by } X_r = H \cap [0, \bigvee X].$$

is a weak ideal system which is a lifting of L.

Proof. We check that r satisfies conditions $(w1)$ to $(w4)$ of Definition 2.1. Let $X \subseteq Y \subseteq H$. For $h \in H$ and $x \in X$, we have $hx \le x \le \bigvee X$, so $hx \in X_r$, thus $(w1)$ holds. Since $X \subseteq Y \subseteq H$, we have $\bigvee X \le \bigvee Y$, so

$$X_r = H \cap [0, \bigvee X] \subseteq H \cap [0, \bigvee Y] = Y_r$$

thus $(w2)$ holds. As H generates L, we have

$$\bigvee X_r = \bigvee (H \cap [0, \bigvee X]) = \bigvee X$$

so

$$(X_r)_r = H \cap [0, \bigvee X_r] = H \cap [0, \bigvee X] = X_r$$

thus $(w3)$ holds. For $c \in H$ we have

$$cX_r = c(H \cap [0, \bigvee X]) \subseteq H \cap [0, \bigvee cX] = (cX)_r$$

so condition $(w4)$ holds. We show that L is isomorphic to the lattice of r-ideals $I_r(H)$. Consider the maps

$$f : I_r(H) \to L \quad \text{given by} \quad X \mapsto \bigvee X$$

and

$$g : L \to I_r(H) \quad \text{given by} \quad y \mapsto H \cap [0, y].$$

As L is generated by H, we have $\bigvee g(y) = y$, so $g(y)$ is indeed an r-ideal of H.

For $X \in I_r(H)$, we have

$$(gf)(X) = H \cap [0, \bigvee X] = X_r = X.$$

Also, for $y \in L$, we have

$$(fg)(y) = \bigvee (H \cap [0, y]) = y$$

as noticed above. Hence f and g are inverse to each other. For $X, Y \in W$, we have

$$f((XY)_r) = \bigvee (XY)_r = \bigvee (XY) = (\bigvee X)(\bigvee Y) = f(X)f(Y)$$

so f is a monoid morphism. If $X, Y \in I_r(H)$ and $X \subseteq Y$, then

$$f(X) = \bigvee X \le \bigvee Y = f(Y).$$

Conversely, if $x, y \in L$ and $x \le y$, then

$$g(x) = H \cap [0, x] \subseteq H \cap [0, y] = g(y).$$

Hence f and g are increasing maps. Thus f is a lattice isomorphism. $\square$

Corollary 2.5. *Under the assumptions of Theorem 2.4, we have*
(i) *r is an ideal system iff H is an M-wire.*
(ii) *r is finitary iff H consists of compact elements.*

Proof. (i) implication ($\Leftarrow$). Let $h, c \in H$ such that $h \le c(\bigvee X)$. As H is an M-wire, we get $h = ck$ for some $k \in H$ with $k \le \bigvee X$. Since L is generated by H, we have

$$(cX)_r = H \cap [0, \bigvee cX] = H \cap [0, c(\bigvee X)] \subseteq c(H \cap [0, \bigvee X]) = cX_r$$

so ($s4$) holds. Thus r is an ideal system.

(i) implication ($\Rightarrow$). Suppose that $s \le ta$ with $s, t \in H$ and $a \in L$. Since H generates L, $a = \bigvee X$ for some $X \subseteq H$. Then $s \in (tX)_r = tX_r$, so $s = tu$ for some $u \in H$ with $u \le a$.

(ii) implication ($\Leftarrow$). Let $X \subseteq H$ and $a \in X_r$; so $a \le \bigvee X$. As a is compact we get $a \le \bigvee Z$ for some finite subset of Z of X. Thus $a \in Z_r$, so r is finitary.

(ii) implication ($\Rightarrow$). Let $s \in H$ and $\{a_\alpha\}_{\alpha \in I} \subseteq L$ such that $s \le \bigvee_{\alpha \in I} a_\alpha$. Write $a_\alpha = \bigvee X_\alpha$ with $X_\alpha \subseteq H$. Then $s \in (\bigcup_{\alpha \in I} X_\alpha)_r$, so $s \in (\bigcup_{\alpha \in J} X_\alpha)_r$ for some finite subset $J \subseteq I$ since r is finitary. We get $s \le \bigvee_{\alpha \in J} a_\alpha$ so s is compact. □

Example 2.6. *Consider the lattice $L = \{0, 1, a, b, c, d\}$ ordered by $a \le b \le d$ and $a \le c \le d$ with multiplication*

$$xy = 0 \textit{ for all } x, y \in \{a, b, c, d\}.$$

It's easy to check that $H = \{0, a, b, c, 1\}$ is a wire, so L lifts to a weak ideal system r whose r-ideals are

$$\{0\}, \{0, a\}, \{0, a, b\}, \{0, a, c\}, \{0, a, b, c\}, \{0, a, b, c, 1\}$$

cf. Theorem 2.4. As the weak meet elements are 0, a and 1 is not liftable to an ideal system, cf. Proposition 2.7 (ii).

A lattice L is called a *domain lattice* if $ab = 0$ with $a, b \in L$ implies $a = 0$ or $b = 0$.

Proposition 2.7. *The following assertions are true.*
(i) *Any lattice can be lifted to a weak ideal system.*
(ii) *A lattice which is liftable to an ideal system is generated by meet principal elements.*
(iii) *A lattice domain which is generated by principal elements is liftable to an ideal system.*

Proof. Let L be a lattice. (i) follows by applying Theorem 2.4 for $H = L$.

(ii) Suppose that L is liftable to an ideal system r on a monoid H. Since the principal r-ideals $aH = \{a\}_r$, $a \in H$, generate $I_r(H)$ it suffices to show that each aH is a meet principal element of $I_r(H)$. Indeed, if A, B are r-ideals, we have the obvious equality $A \cap Ba = a((A : a) \cap B)$.

(iii) Suppose that L is generated by its subset H of principal elements. By [4, Corollary 3.3], H is a submonoid of L, so H is a wire. Suppose that $s \le ta$ with $s, t \in H$ and $a \in L$. As t is principal, $s = tu$ for some $u \le a$ in H cf. [2, Theorem 7]. So H is an M-wire, hence L is liftable to an ideal system cf. Theorem 2.4. □

Example 2.8. *Consider the lattice $\mathbb{N}$ with usual number multiplication where $\bigvee = gcd$ and $\bigwedge = lcm$. Let D be a ring of algebraic integers. Sending each $X \subseteq D$ into X_r = the ideal generated by X, we get an ideal system whose r-ideal lattice is the usual ideal lattice I_D of D. Note that the set of principal ideals of D is an M-wire of I_D. It is well-known that I_D is isomorphic to $\mathbb{N}$. So the lattice $\mathbb{N}$ can be lifted to an ideal system in infinitely many ways and it has infinitely many M-wires. Our next result explores some M-wires of $\mathbb{N}$ given by the norm function on a ring of quadratic integers.*

Let D be a nonfactorial ring of quadratic integers with class group G and let $N : D \to \mathbb{N}$ the absolute value of its norm function. Let S be the multiplicatively closed subset of $\mathbb{N}$ generated by the image $Im(N)$ of N and the set I of all prime numbers which are inert in D.

Proposition 2.9. *With notation above, S is an M-wire on the lattice $\mathbb{N}$ (see Example 2.8) iff G is a finite product of copies of $\mathbb{Z}_2$.*

Proof. We shall use repeatedly the following well-known Number Theory facts: D is a Dedekind domain, G is finite and every class $g \in G$ contains infinitely many prime ideals of D. We first prove that S generates $\mathbb{N}$ as a lattice (i.e. S is a wire). It suffices to show that every prime number $p \in \mathbb{N} - I$ is the gcd of some numbers in S. Take a prime ideal P of norm p. If P is principal, then $p \in S$. Suppose that P is not principal and let e be its class in G. Inside $-e$ take another two prime ideals Q and R of norms q and r respectively. We may arrange that p, q, r are distinct. As PQ and PR are principal ideals, we get $pq, pr \in S$ and p is their gcd.

Therefore, we may assume from the very beginning that S is a wire. It's easy to see that S is an M-wire iff S is closed under division iff $Im(N)$ is closed under division (i.e. if $a, b \in Im(N) - \{0\}$ and $a|b$, then $b/a \in Im(N)$). So it remains to show that $Im(N)$ is closed under division iff G is a finite product of copies of $\mathbb{Z}_2$.

Suppose that $Im(N)$ is closed under division. Then G has no odd order element. Deny. Let P be a prime ideal of D whose ideal class has odd order m and let p be the norm of P. Then $p^m \in Im(N)$ and, since p^2 is clearly in $Im(N)$, we get that $p \in Im(N)$, as $Im(N)$ is closed under division. But this is a contradiction because P is not principal. To show that G is a finite product of copies of $\mathbb{Z}_2$, it suffices to prove that G has no element of order four. Deny. Let $g \in G$ of order four. Select prime ideals P, Q of norms p, q in classes $g, 2g$ respectively. Denote the conjugate of P by $\overline{P}$. Since $P\overline{P}$ and P^2Q are principal ideals, we get that p^2 and p^2q are in $Im(N)$, so $q \in Im(N)$, as $Im(N)$ is closed under division. But this is a contradiction because Q is not principal.

Conversely, suppose that G is a finite product of copies of $\mathbb{Z}_2$. Let $a, b \in D - \{0\}$ such that $N(a) \mid N(b)$. Since D is a Dedekind domain, we may consider the prime power factorizations $aD = P_1 \cdots P_n$ and $bD = P_{n+1} \cdots P_m$. We use now the fact that each element in G has order ≤ 2. We replace some of the factors P_i by their conjugate such that finally no pair of distinct conjugates appears in list $\{P_1, ..., P_m\}$. Doing this we change a and b but we preserve their norm. Moreover, in the new setup it follows that a divides b, so $N(b)/N(a) = N(b/a) \in Im(N)$. □

Remark 2.10. *With notation above, S is an M-wire on the lattice $\mathbb{N}$ provided $D = \mathbb{Z}[\sqrt{-5}]$ (since its class group is $\mathbb{Z}_2$) but S is not an M-wire on the lattice $\mathbb{N}$ if $D = \mathbb{Z}[\sqrt{-17}]$ since* *its class group is* $\mathbb{Z}_4$ *or* $N(5+\sqrt{-17}) = 42$, $N(2+\sqrt{-17}) = 21$ *but there is not a single element in* $\mathbb{Z}[\sqrt{-17}]$ *of norm 2.*

We put Theorem 2.4 to work. Let r be a weak ideal system on a monoid H. Recall that the map $r_s : \mathcal{P}(H) \to \mathcal{P}(H)$ given by

$$X_{r_s} = \bigcup \{Z_r | Z \text{ finite subset of } X\} \text{ for all } X \subseteq H$$

is a finitary weak ideal system called the finitary weak ideal system associated to r. See [5, Chapter 3] for details.

Remark 2.11. *We give the following application of Theorem 2.4. To a lattice* L *we can canonically associate a lattice* L' *generated by compact elements. Let* r *be the ideal system on* H *constructed in Theorem 2.4 for* $H = L$. *Let* r_s *be the finitary weak ideal system associated to* r *recalled above. By Theorem 2.2, the lattice* $L' = I_{r_s}(H)$ *of all* r_s*-ideals of* H *is generated by compact elements. By Theorem 2.4 and the definition of* r_s *we get*

$$X_{r_s} = \{h \in L \mid h \leq h_1 \vee ... \vee h_n \textit{ for some } h_1, ..., h_n \in X\}.$$

We get the set embedding

$$L \to L', \quad x \mapsto [0, x]$$

which is a lattice isomorphism when L *is generated by compact elements, because in that case* $r = r_s$.

Example 2.12. *As an illustration of Remark 2.11 consider the lattice* $L = [0, 1] \subseteq \mathbb{R}$ *with usual number multiplication with* $\bigvee = sup$ *and* $\bigwedge = inf$. *No nonzero element* x *of* L *is compact because*

$$x = \bigvee \{x - 1/n \mid n \geq 1/x, \ n \in \mathbb{N}\}$$

but any finite subjoin is $< x$. *Performing the construction in Remark 2.11 we get the lattice*

$$L' = A \cup B \textit{ with } A = \{ \ [0, x] \mid x \in [0, 1]\} \textit{ and } B = \{ \ [0, x) \mid x \in [0, 1]\}.$$

The multiplication in L' *is the usual interval multiplication. For* $X \subseteq L$ *with* $a = sup(X)$, *we get that* X_{r_s} *is* $[0, a]$ *resp.* $[0, a)$ *if* $a \in X$ *resp.* $a \notin X$. *Each element of* A *is compact in* L'.

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