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Homological shifts of polymatroidal ideals

by

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Abstract

We study the homological shifts of polymatroidal ideals. In our main theorem we prove that the first homological shift ideal of any polymatroidal ideal is again polymatroidal, supporting a conjecture of Bandari, Bayati and Herzog that predicts that all homological shift ideals of a polymatroidal ideal are polymatroidal. As a nice consequence, we recover a result of Bayati which proves this conjecture in the squarefree case.

Key Words: Monomial ideals, minimal resolution, multigraded shifts, polymatroidal ideals.

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1 Introduction

The study of the minimal free resolutions of monomial ideals is a main topic in combinatorial commutative algebra. Let K be a field, I a monomial ideal of the standard graded polynomial ring $S = K[x_1, \dots, x_n]$ and $\mathbb{F}$ be its minimal multigraded free S-resolution. Then, the free modules in the resolution $\mathbb{F}$ are of the form $F_i = \bigoplus_{j=1}^{\beta_i(I)} S(-\mathbf{a}_{i,j})$, where $\mathbf{a}_{i,j}$ are integral vectors with non negative entries, called the *multigraded shifts* of I. For an integral vector $\mathbf{a} = (a_1, a_2, \dots, a_n)$, we let $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Using this notation, the *ith homological shift ideal* of I is defined as $\mathrm{HS}_i(I) = (\mathbf{x}^{\mathbf{a}_{i,j}} : j = 1, \dots, \beta_i(I))$, see [15]. Note that $\mathrm{HS}_0(I) = I$ and $\mathrm{HS}_j(I) = (0)$ for $j > \mathrm{pd}(I)$.

The basic goal of this theory is to understand how much of the homological and combinatorial properties of a monomial ideal I are inherited by its homological shift ideals. We refer to any combinatorial or homological property enjoyed by $\mathrm{HS}_0(I) = I$ and by $\mathrm{HS}_j(I)$, for all $1 \le j \le \mathrm{pd}(I)$, as an *homological shift property* of I. This rather new trend of research, see [3, 4, 5, 6, 7, 8, 9, 15, 16], has its origins in a meeting between Somayeh Bandari, Shamila Bayati and Jürgen Herzog that took place in Essen in 2012. Due to experimental evidence, the three authors conjectured that the property of being *polymatroidal* is an homological shift property. For a monomial u, we define the x_i*-degree* of u as $\deg_{x_i}(u) = \max\{j : x_i^j \text{ divides } u\}$. Recall that a *polymatroidal* ideal I is an equigenerated monomial ideal of S whose minimal generating set $G(I)$ corresponds to the base of a *discrete polymatroid*. The bases of a discrete polymatroidal can be characterized in term of the so–called *exchange property*. Thus an ideal $I \subset S$ is polymatroidal if and only if the following exchange property holds: for all $u, v \in G(I)$ such that $\deg_{x_i}(u) > \deg_{x_i}(v)$, there exists j such that $\deg_{x_j}(u) < \deg_{x_j}(v)$ and $x_j(u/x_i) \in G(I)$. The Bandari–Bayati–Herzog conjecture is widely open. Until now it is solved only in two cases: for squarefree polymatroidal, *i.e.*, *matroidal ideals* [3], and for polymatroidal ideals that satisfy the *strong exchange property* [15].

Polymatroidal ideals are one of the most distinguished classes of monomial ideals. Indeed, the product of polymatroidal ideals is polymatroidal. Any polymatroidal ideal has linear quotients and thus a linear resolution. Thus, they have *linear powers*, [12, Corollary 12.6.4], a rare property among monomial ideals. In this paper our aim is to study the homological shift ideals of polymatroidal ideals. See also [1, 2, 11, 13, 17, 19, 20].

The paper is organized as follows. In Section 2, we collect some basic facts on homological shift ideals of equigenerated monomial ideals with linear quotients. Recall that a monomial ideal $I \subset S$ has *linear quotients* if for some *admissible order* $u_1 > \cdots > u_m$ of its minimal generating set $G(I)$, the colon ideals $(u_1, \ldots, u_{k-1}) : u_k$ are generated by a subset of the variables, for $k = 2, \ldots, m$. Quite generally, to determine the ideals $\mathrm{HS}_j(I)$ for a monomial ideal I is difficult. Nonetheless, this is easy for ideals with linear quotients as shown in Proposition 1.

In Section 3, we prove our main theorem. We are able to show that $\mathrm{HS}_1(I)$ is polymatroidal if I is polymatroidal (Theorem 1). This is the first result valid for *all* polymatroidal ideals, that supports Conjecture 1. Our proof is based on Proposition 2, whose main advantage consists in the fact that $\mathrm{HS}_1(I)$ does not depend upon the admissible order of I. To study the higher homological shift ideals, firstly we note that for $j \geq 1$, $\mathrm{HS}_{j+1}(I) \subseteq (\mathrm{HS}_1(\mathrm{HS}_j(I)))_{>j+1}$, (Corollary 1), where $J_{>j+1}$ is the monomial ideal with $G(J_{>j+1}) = \{u \in G(J) : |\mathrm{supp}(u)| > j+1\}$ as a minimal generating set and $\mathrm{supp}(u) = \{i : x_i \text{ divides } u\}$. Unfortunately, equality in the above inclusion does not hold in general, (Example 1). Nonetheless, it holds for matroidal ideals, (Proposition 4). As a consequence Conjecture 1 holds for all matroidal ideals, (Corollary 2). It would be of interest to classify all polymatroidal ideals satisfying the equation $\mathrm{HS}_{j+1}(I) = (\mathrm{HS}_1(\mathrm{HS}_j(I)))_{>j+1}$ for all $j < \mathrm{pd}(I)$.

2 Generalities on homological shift ideals

Let $S = K[x_1, \ldots, x_n]$ be the standard graded polynomial ring over a field K. For a monomial $u = x_1^{a_1} \cdots x_n^{a_n} \in S$, the vector $\mathbf{a} = (a_1, \ldots, a_n)$ is called the *multidegree* of u. We put $u = \mathbf{x}^{\mathbf{a}}$. For $\mathbf{a} = \mathbf{0} = (0, 0, \ldots, 0)$, $\mathbf{x}^{\mathbf{0}} = 1$. Whereas $\deg(u) = a_1 + a_2 + \cdots + a_n$ is the *degree* of u. Let $G(I)$ be the unique minimal monomial generating set of I, and $G(I)_d = \{u \in G(I) : \deg(u) = d\}$.

Definition 1. Let $I \subset S$ be a monomial ideal, and let $(\mathbb{F}, \partial)$ be the minimal multigraded free resolution of I. The *ith homological shift ideal* of I is defined as

$$\mathrm{HS}_i(I) \;=\; (\mathbf{x}^{\mathbf{a}} \;:\; \beta_{i,\mathbf{a}}(I) \neq 0).$$

Here $\beta_{i,\mathbf{a}}(I)$ is a multigraded Betti number.

Clearly $\mathrm{HS}_0(I) = I$ and $\mathrm{HS}_j(I) = (0)$ for $j > \mathrm{pd}(I)$. In general, we ask what properties of I are inherited by its homological shift ideals.

Let I be a monomial ideal with minimal generating set $G(I) = \{u_1, \ldots, u_m\}$. We say that I has *linear quotients* with respect to the order $u_1 > \cdots > u_m$ of its minimal generators, if for all $k = 2, \dots, m$, the colon ideal $(u_1, \dots, u_{k-1}) : u_k$ is generated by a subset of the variables, $x_1, \dots, x_n$. Any such ordering of $G(I)$ is called an *admissible order* of I. Set

$$\text{set}(u_k) \ = \ \{i : x_i \in (u_1, \dots, u_{k-1}) : u_k\},$$

for $k = 2, \dots, m$ and $\text{set}(u_1) = \emptyset$. Furthermore, put $\mathbf{x}_F = \prod_{i \in F} x_i$ if F is non empty, and $\mathbf{x}_\emptyset = 1$. By [18, Lemma 1.5], we have the following result.

Proposition 1. *Let $I \subset S$ be a monomial ideal with linear quotients with admissible order $u_1 > \dots > u_m$ of $G(I)$. Then,*

$$\text{HS}_j(I) \ = \ (\mathbf{x}_F u \ : \ u \in G(I), \ F \subseteq \text{set}(u), \ |F| = j). \tag{2.1}$$

Let $u = \mathbf{x}^{\mathbf{a}} \in S$, the x_i*-degree* of u is $\deg_{x_i}(u) = \max\{j : x_i^j \text{ divides } u\} = a_i$. Given monomials of the same degree $u, v \in S$, the *distance* between u and v is

$$d(u, v) \ = \ \frac{1}{2} \sum_{i=1}^{n} \big| \deg_{x_i}(u) - \deg_{x_i}(v) \big|.$$

Lemma 1. *Let u, v monomials of S of the same degree. Then, $u = x_k(v/x_\ell)$ for some $k \neq \ell$, if and only if $d(u, v) = 1$.*

Proof. Let $v = x_1^{b_1} \cdots x_n^{b_n}$, then $u = x_k(v/x_\ell) = (\prod_{i \neq k, \ell} x_i^{b_i}) x_k^{b_k+1} x_\ell^{b_\ell - 1}$. Note that

$$\big| \deg_{x_i}(u) - \deg_{x_i}(v) \big| \ = \ \begin{cases} |b_\ell - 1 - b_\ell| & \text{if } i = \ell, \\ |b_k + 1 - b_k| & \text{if } i = k, \\ \quad 0 & \text{otherwise.} \end{cases}$$

Thus $d(u, v) = \frac{1}{2} \sum_i |\deg_{x_i}(u) - \deg_{x_i}(v)| = \frac{1}{2}\big(|b_k + 1 - b_k| + |b_\ell - 1 - b_\ell|\big) = \frac{1}{2}(1 + 1) = 1$.

Conversely, assume $d(u, v) = 1$. From the definition of $d(u, v)$ it is clear that either $u = x_k^2 v$ or $u = x_k(v/x_\ell)$, for some $k \neq \ell$. But the first possibility does not occur, lest $\deg(u) > \deg(v)$. Therefore, the desired conclusion follows. ∎

Combining Lemma 1 with [15, Proposition 1.3] we have

Proposition 2. *Let $I \subset S$ be an equigenerated monomial ideal with linear quotients. Then*

$$\text{HS}_1(I) \ = \ (\text{lcm}(u, v) \ : \ u, v \in G(I), \ d(u, v) = 1).$$

Let $\succ$ be a *monomial order* on S. Up to a relabeling on the variables, we may assume that $x_1 \succ \dots \succ x_n$. We say that $\succ$ *is induced by* $x_1 > \dots > x_n$. We say that I has linear quotients with respect to $\succ$ if I has linear quotients with admissible order $u_1 \succ u_2 \succ \dots \succ u_m$ of $G(I)$. A particular monomial order is the *lex order* $>_{\text{lex}}$ induced by $x_1 > \dots > x_n$. Let $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}$ be monomials of S. Then $\mathbf{x}^{\mathbf{a}} >_{\text{lex}} \mathbf{x}^{\mathbf{b}}$ if $a_1 = b_1, \dots, a_{s-1} = b_{s-1}$ and $a_s > b_s$, for some $1 \leq s \leq n$.

We set $[n] = \{1, 2, \dots, n\}$. For a monomial $u \in S$, we define its *support* as $\text{supp}(u) = \{i \in [n] : \deg_{x_i}(u) > 0\}$ and we set $\max(u) = \max \text{supp}(u)$.

Lemma 2. *Let $I \subset S$ be an equigenerated monomial ideal with linear quotients with respect to $\succ$ induced by $x_1 > \cdots > x_n$. Then, for all $u \in G(I)$,*

$$\mathrm{set}(u) \ \subseteq\ [\max(u) - 1].$$

Proof. Indeed, let $G(I)$ ordered as $u_1 \succ \cdots \succ u_m$ and let $j \in \{1, \dots, m\}$. If $i \in \mathrm{set}(u_j)$, then $x_i u_j \in (u_1, \dots, u_{j-1})$. Since $\deg(u_1) = \cdots = \deg(u_{j-1}) = \deg(u_j)$, there exists $s \in \mathrm{supp}(u_j)$, $s \neq i$ such that $x_i(u_j/x_s) = u_p$ for some $p \leq j-1$. But $x_i(u_j/x_s) = u_p \succ u_j$. Since $\succ$ is a monomial order, this implies that $x_i u_j \succ x_s u_j$, thus $x_i \succ x_s$. Hence $i < s$. But $s \leq \max(u_j)$ and so $i < \max(u_j)$. ∎

In [15], the following general inclusion was shown.

Proposition 3. [15, Proposition 1.4] *Let $I \subset S$ be a monomial ideal with linear quotients. Then $\mathrm{HS}_{j+1}(I) \subseteq \mathrm{HS}_1(\mathrm{HS}_j(I))$, for all j.*

However, in general $\mathrm{HS}_{j+1}(I) \neq \mathrm{HS}_1(\mathrm{HS}_j(I))$. Let $I = (x_2x_4, x_1x_2, x_1x_3)$. Then I has linear quotients with admissible order $x_2x_4 > x_1x_2 > x_1x_3$. We have $\mathrm{HS}_1(I) = (x_1x_2x_3, x_1x_2x_4)$ and $\mathrm{HS}_2(I) = (0)$, but $\mathrm{HS}_1(\mathrm{HS}_1(I)) = (x_1x_2x_3x_4) \neq (0) = \mathrm{HS}_2(I)$.

For a monomial ideal $J \subset S$, let $J_{>\ell}$ be the monomial ideal whose minimal generating set is $G(J_{>\ell}) = \{u \in G(J) : |\mathrm{supp}(u)| > \ell\}$.

Corollary 1. *Let $I \subset S$ be an equigenerated monomial ideal with linear quotients with respect to a monomial order $\succ$ (e.g., $>_{\mathrm{lex}}$) induced by $x_1 > \cdots > x_n$. Then,*

$$\mathrm{HS}_{j+1}(I) \ \subseteq\ \big(\mathrm{HS}_1(\mathrm{HS}_j(I))\big)_{>j+1}.$$

Proof. For $j = 0$ the assertion is immediate. Let $j > 0$. Firstly we show the inclusion $\mathrm{HS}_{j+1}(I) \subseteq \mathrm{HS}_1(\mathrm{HS}_j(I))$. By Proposition 2,

$$\mathrm{HS}_1(\mathrm{HS}_j(I)) = (\mathrm{lcm}(w_1, w_2)\ :\ w_1, w_2 \in G(\mathrm{HS}_j(I)),\ d(w_1, w_2) = 1).$$

Take $w = \mathbf{x}_F u \in G(\mathrm{HS}_{j+1}(I))$ with $u \in G(I)$, $F \subseteq \mathrm{set}(u)$ and $|F| = j+1$. Since $j + 1 \geq 2$ we can find $r, s \in F$, $r \neq s$. Then $w_1 = \mathbf{x}_{F\setminus\{r\}}u, w_2 = \mathbf{x}_{F\setminus\{s\}}u \in G(\mathrm{HS}_j(I))$ and $d(w_1, w_2) = 1$. Thus $\mathrm{lcm}(w_1, w_2) = w \in G(\mathrm{HS}_1(\mathrm{HS}_j(I)))$, as desired. It remains to prove that any $w = \mathbf{x}_F u \in G(\mathrm{HS}_{j+1}(I))$ has $\mathrm{supp}(w) > j + 1$. By Lemma 2 we have $F \subseteq \mathrm{set}(u) \subseteq [\max(u) - 1]$. Hence $\max(u) \notin F$ and $F \cup \{\max(u)\} \subseteq \mathrm{supp}(w)$, and since $|F| = j + 1$ we have that $\mathrm{supp}(w) \geq |F| + 1 = j + 2$, as desired. ∎

3 The first homological shift ideal of polymatroidal ideals

Recall that an equigenerated monomial ideal $I \subset S$ is a *polymatroidal* ideal if it satisfies the following *exchange property*,

(∗) for all $u, v \in G(I)$ and all i such that $\deg_{x_i}(u) > \deg_{x_i}(v)$, there exists j with $\deg_{x_j}(u) < \deg_{x_j}(v)$ and such that $x_j(u/x_i) \in G(I)$.

Such ideals are called *polymatroidal* because their minimal generating set $G(I)$ corresponds to the basis of a *discrete polymatroid*, see [12, Chapter 12] for more information on this subject. For later use, we recall that a polymatroidal ideal satisfies also the following *dual exchange property*, [11, Lemma 2.1],

(∗∗) for all $u, v \in G(I)$ and all j such that $\deg_{x_j}(u) < \deg_{x_j}(v)$, there exists i with $\deg_{x_i}(u) > \deg_{x_i}(v)$ and such that $x_j(u/x_i) \in G(I)$.

It is expected that the following is true.

Conjecture 1. (Bandari–Bayati–Herzog), [3, 15]. *Let $I \subset S$ be a polymatroidal ideal. Then all homological shift ideals $\mathrm{HS}_j(I)$ are again polymatroidal, for all $j \geq 0$.*

It is known that a polymatroidal ideal $I \subset S$ has linear quotients with respect to the lex order induced by $x_1 > \cdots > x_n$, see [2, Theorem 2.4]. Using the description of $\mathrm{HS}_1(I)$ given in Proposition 2 we can prove

Theorem 1. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a polymatroidal ideal. Then $\mathrm{HS}_1(I)$ is a polymatroidal ideal.*

Proof. By Proposition 2,

$$\begin{aligned}\mathrm{HS}_1(I) &= (x_i u \ : \ u \in G(I),\ i \in \mathrm{set}(u)) \\ &= (\mathrm{lcm}(u, v) \ : \ u, v \in G(I),\ d(u, v) = 1).\end{aligned}$$

We must prove the following exchange property,

(∗) for all monomials $u, v \in G(I)$, all integers $k \in \mathrm{set}(u)$, $\ell \in \mathrm{set}(v)$ such that $u_1 = x_k u \neq x_\ell v = v_1$ and all i with $\deg_{x_i}(u_1) > \deg_{x_i}(v_1)$, there exists j such that $\deg_{x_j}(u_1) < \deg_{x_j}(v_1)$ and $x_j(u_1/x_i) = x_j(x_k u)/x_i \in G(\mathrm{HS}_1(I))$.

We may assume that i is different both from k and ℓ. Indeed, if $k = i$, then as $k \in \mathrm{set}(u)$ we have $u_1 = x_p z$ for some $z \in G(I) \setminus \{u\}$, $p \neq k$, and we may use the element $x_p z$ with $p \neq k = i$. The same reasoning applies for ℓ. In particular, since $i \neq k$, $i \neq \ell$ and by hypothesis $\deg_{x_i}(u_1) > \deg_{x_i}(v_1)$, we have $\deg_{x_i}(u) > \deg_{x_i}(v)$ as well. Since I is polymatroidal, the set

$$\Omega \ = \ \{h \in [n] \setminus \{i\} \ : \ \deg_{x_h}(u) < \deg_{x_h}(v) \text{ and } x_h(u/x_i) \in G(I)\}.$$

is non empty. Let $h \in \Omega$ and set $w = x_h(u/x_i)$. We distinguish two cases.

Case 1. Suppose that $k \in \Omega$. For $h = k \in \Omega$, we have $\deg_{x_k}(u) < \deg_{x_k}(v)$ and $w = x_k(u/x_i) \in G(I)$. We distinguish two more cases.

Subcase 1.1. Assume $w = v$. By hypothesis $v_1 = x_\ell v \in G(\mathrm{HS}_1(I))$. We show that the property (∗) is verified for the integer $j = \ell$. Indeed, as $i \neq \ell$ and $w = v$,

$$\begin{aligned}\deg_{x_\ell}(u_1) = \deg_{x_\ell}(x_k u) &= \deg_{x_\ell}(x_k u/x_i) = \deg_{x_\ell}(w) \\ &= \deg_{x_\ell}(v) < \deg_{x_\ell}(x_\ell v),\end{aligned}$$

and $v_1 = x_\ell v = x_\ell w = x_\ell(x_k u)/x_i \in G(\mathrm{HS}_1(I))$, as desired.

Subcase 1.2. Assume $w \neq v$. Thus, for some r, $\deg_{x_r}(w) > \deg_{x_r}(v)$. Since I is polymatroidal, there exists an integer m with $\deg_{x_m}(w) < \deg_{x_m}(v)$ and such that $w_1 = x_m(w/x_r) \in G(I)$. Clearly $m \neq r$. Hence, Lemma 1 implies that $d(w, w_1) = 1$ and Proposition 2 implies that

$$\mathrm{lcm}(w, w_1) = x_m w = x_m(x_\ell u)/x_i \in G(\mathrm{HS}_1(I)). \tag{3.1}$$

It remains to prove that the integer m satisfies the first condition of property $(*)$, namely $\deg_{x_m}(u_1) < \deg_{x_m}(v_1)$. First note that $m \neq i$. Lest, if $i = m$, by hypothesis $\deg_{x_i}(w) < \deg_{x_i}(v)$ and then $\deg_{x_i}(u) = \deg_{x_i}(w) + 1 \leq \deg_{x_i}(v)$, against the fact that $\deg_{x_i}(u) > \deg_{x_i}(v)$. Thus, since $m \neq i$, we have $\deg_{x_m}(u) \leq \deg_{x_m}(w) < \deg_{x_m}(v)$. This inequality together with equation (3.1) show that the integer $j = m$ satisfies the property $(*)$ in such a case.

Case 2. Suppose that $k \notin \Omega$. Nonetheless, for some $h \in \Omega$, with $h \neq k$, we have $\deg_{x_h}(u) < \deg_{x_h}(v)$ and $w = x_h(u/x_i) \in G(I)$. Since $k \in \mathrm{set}(u)$, there exist $z \in G(I)$, $z \neq u$ and $x_p \neq x_k$ such that $u_1 = x_k u = x_p z$.

Subcase 2.1. Suppose $d(w, z) = 1$. As $h \in \Omega$ but $k \notin \Omega$ we have $h \neq k$. Thus, as $w = x_h(u/x_i)$ and $z = x_k(u/x_p)$ it follows that $p = i$, lest $d(w, z) > 1$. Hence $p = i$ and Proposition 2 implies

$$\mathrm{lcm}(w, z) = \mathrm{lcm}(x_h(u/x_i), x_k(u/x_p)) = x_h(x_k u)/x_i \in G(\mathrm{HS}_1(I)).$$

Finally, we just need to check that $\deg_{x_h}(u_1) < \deg_{x_h}(v_1)$. Indeed, as $h \neq k$,

$$\deg_{x_h}(x_k u) = \deg_{x_h}(u) < \deg_{x_h}(v) \leq \deg_{x_h}(x_\ell v).$$

Subcase 2.2. Suppose $d(w, z) > 1$. Then $p \neq i$, lest $d(w, z) = 1$ by Subcase 2.1. Thus $d(w, z) = d(x_h(u/x_i), x_k(u/x_p)) = 2$, $i \neq h$, $h \neq k$, $k \neq p$, $p \neq i$ and

$$\begin{aligned} &\deg_{x_i}(w) < \deg_{x_i}(z), &\qquad &\deg_{x_h}(w) > \deg_{x_h}(z),\\ &\deg_{x_k}(w) < \deg_{x_k}(z), &\qquad &\deg_{x_p}(w) > \deg_{x_p}(z). \end{aligned}$$

Moreover, for all $q \neq i, h, k, p$ we have $\deg_{x_q}(w) = \deg_{x_q}(z)$. Since $w, z \in G(I)$ and $\deg_{x_i}(z) > \deg_{x_i}(w)$ we have $z_1 = x_h(z/x_i) \in G(I)$ or $z_2 = x_p(z/x_i) \in G(I)$. We distinguish two more cases.

Subcase 2.2.1. Suppose $z_1 = x_h(z/x_i) \in G(I)$. Note that

$$x_p(z_1/x_k) = x_p(x_h(z/x_i))/x_k = x_p x_h x_k((u/x_p)/x_i)/x_k = x_h(u/x_i) = w.$$

Since $k \neq p$, Lemma 1 implies that $d(z_1, w) = 1$. Thus, by Proposition 2

$$\begin{aligned} \mathrm{lcm}(z_1, w) &= \mathrm{lcm}(x_h(z/x_i), x_h(u/x_i))\\ &= \mathrm{lcm}(x_h(x_k(u/x_p)/x_i), x_h(u/x_i))\\ &= x_h(x_k u)/x_i \in G(\mathrm{HS}_1(I)), \end{aligned}$$

and the property $(*)$ is satisfied as $h \in \Omega$, that is $\deg_{x_h}(u) < \deg_{x_h}(v)$ and as $h \neq k$, we have $\deg_{x_h}(u_1) = \deg_{x_h}(x_k u) = \deg_{x_h}(u) < \deg_{x_h}(v) \leq \deg_{x_h}(x_\ell v) = \deg_{x_h}(v_1)$.

Subcase 2.2.2. Suppose $z_2 = x_p(z/x_i) \in G(I)$. Note that

$$z_2 = x_p(z/x_i) = x_p(x_k(u/x_p)/x_i) = x_k(u/x_i)$$

and $d(z_2, w) = 1$. Thus, Proposition 2 implies that

$$\mathrm{lcm}(z_2, w) = \mathrm{lcm}(x_k(u/x_i), x_h(u/x_i)) = x_h(x_k u)/x_i \in G(\mathrm{HS}_1(I)),$$

and as before $\deg_{x_h}(u_1) < \deg_{x_h}(v_1)$. The proof is complete. ∎

By [2, Theorem 2.4], a polymatroidal ideal $I \subset S$ has linear quotients with respect to the lex order $>_{\mathrm{lex}}$ (induced by any ordering of the variables $x_1, \ldots, x_n$). Thus Corollary 1 implies that for all $j \geq 0$,

$$\mathrm{HS}_{j+1}(I) \ \subseteq \ \big(\mathrm{HS}_1(\mathrm{HS}_j(I))\big)_{>j+1}.$$

Next we study when equality holds. This is the case when I is actually matroidal, that is, a squarefree polymatroidal ideal [3, Corollary 2.3]. The proof in [3] uses matroid and graph theory. We provide a purely algebraic proof. Firstly, we note the following general fact.

Lemma 3. *Let $I \subset S$ be a squarefree ideal. Then, for all $j \geq 0$,*

$$\big(\mathrm{HS}_1(\mathrm{HS}_j(I))\big)_{>j+1} \ = \ \mathrm{HS}_1(\mathrm{HS}_j(I)).$$

Proof. Since I is squarefree, all $\mathrm{HS}_j(I)$ are squarefree. It follows from [14, Lemma 4.4] that all monomials $w \in G(\mathrm{HS}_j(I))$ have $|\mathrm{supp}(w)| > j$.

It is clear that $(\mathrm{HS}_1(\mathrm{HS}_j(I)))_{>j+1} \subseteq \mathrm{HS}_1(\mathrm{HS}_j(I))$. We show the opposite inclusion. Let $y \in G(\mathrm{HS}_1(\mathrm{HS}_j(I)))$. We claim that $|\mathrm{supp}(y)| > j+1$. By Proposition 2, $y = \mathrm{lcm}(w_1, w_2)$ with $w_1, w_2 \in G(\mathrm{HS}_j(I))$ such that $d(w_1, w_2) = 1$. By Lemma 1, $w_1 = x_k(w_2/x_\ell)$ for some $k \neq \ell$. Thus $y = \mathrm{lcm}(w_1, w_2) = x_\ell w_1$. We have shown that $w_1 \in G(\mathrm{HS}_j(I))$ has $|\mathrm{supp}(w_1)| \geq j+1$. Since $\ell \notin \mathrm{supp}(w_1)$, $|\mathrm{supp}(y)| = 1 + |\mathrm{supp}(w_1)| \geq j+2 > j+1$, as desired. ∎

Proposition 4. *Let $I \subset S$ be a matroidal ideal. Then $\mathrm{HS}_{j+1}(I) = \mathrm{HS}_1(\mathrm{HS}_j(I))$ for all $j < \mathrm{pd}(I)$.*

Proof. Since I is squarefree, all homological shift ideals involved in the proof are squarefree. Fix the lex order $>_{\mathrm{lex}}$ induced by $x_1 > \cdots > x_n$. Then I has linear quotients with respect to $>_{\mathrm{lex}}$, [2, Theorem 2.4]. For $u \in G(I)$, we denote by $\mathrm{set}(u)$ the following set $\{i : x_i \in (v \in G(I) : v >_{\mathrm{lex}} u) : u\}$. By equation (2.1),

$$\mathrm{HS}_j(I) \ = \ (\mathbf{x}_F u \ : \ u \in G(I),\ F \subseteq \mathrm{set}(u),\ |F| = j). \tag{3.2}$$

For $j = 0$, there is nothing to prove. Let $1 \le j < \mathrm{pd}(I)$. By Proposition 3 we have $\mathrm{HS}_{j+1}(I) \subseteq \mathrm{HS}_1(\mathrm{HS}_j(I))$. So we only need to prove that $\mathrm{HS}_1(\mathrm{HS}_j(I)) \subseteq \mathrm{HS}_{j+1}(I)$. By Proposition 2, we have

$$\mathrm{HS}_1(\mathrm{HS}_j(I)) = (\mathrm{lcm}(w_1, w_2) \ : \ w_1, w_2 \in G(\mathrm{HS}_j(I)),\ d(w_1, w_2) = 1).$$

Thus, we must show that for all $w_1, w_2 \in G(\mathrm{HS}_j(I))$ with $d(w_1, w_2) = 1$ we have $\mathrm{lcm}(w_1, w_2) \in G(\mathrm{HS}_{j+1}(I))$. By equation (3.2), $w_1 = \mathbf{x}_F u \neq w_2 = \mathbf{x}_G v$ with $u, v \in G(I)$, $F \subseteq \mathrm{set}(u)$, $G \subseteq \mathrm{set}(v)$ and $|F| = |G| = j$. Since $d(w_1, w_2) = 1$, Lemma 1 gives $w_1 = x_k(w_2/x_\ell)$ for some $k \neq \ell$. As observed before, w_1, w_2 are squarefree. Hence, $\mathrm{supp}(w_1) = \{k\} \cup (\mathrm{supp}(w_2) \setminus \{\ell\})$. Note that as $\ell \in \mathrm{supp}(w_2)$, we can find $z \in G(I)$ such that $\ell \in \mathrm{supp}(z)$. Indeed if $\ell \in \mathrm{supp}(v)$ then we can choose $z = v$. Otherwise, $\ell \in G \subseteq \mathrm{set}(v)$, and $x_\ell v = x_s z$ with $z \in G(I) \setminus \{v\}$, $s \neq \ell$ and so $\ell \in \mathrm{supp}(z)$.

Since $\ell \in \mathrm{supp}(z) \setminus \mathrm{supp}(u)$ it is $\deg_{x_\ell}(z) > \deg_{x_\ell}(u)$. By the dual exchange property $(**)$, the set of integers h such that $\deg_{x_h}(u) > \deg_{x_h}(z)$ and $x_\ell(u/x_h) \in G(I)$ is non empty. Hence, the following set is non empty too

$$\Omega \ = \ \{h \in [n] \setminus \{\ell\} \ : \ x_\ell(u/x_h) \in G(I)\}.$$

Case 1. Assume there exists $h \in \Omega$ with $h > \ell$. Then $x_\ell(u/x_h) >_{\mathrm{lex}} u$ and $\ell \in \mathrm{set}(u)$. Now $\mathrm{lcm}(w_1, w_2) = \mathrm{lcm}(w_1, x_\ell(w_1/x_k)) = x_\ell w_1 = x_\ell \mathbf{x}_F u$. Since $\ell \notin \mathrm{supp}(w_1)$, we also have $\ell \notin F$. Since $\ell \in \mathrm{set}(u)$, we have that $F \cup \{\ell\}$ is a subset of $\mathrm{set}(u)$ having cardinality $j + 1$ and $\mathrm{lcm}(w_1, w_2) = \mathbf{x}_{F\cup\{\ell\}} u \in G(\mathrm{HS}_{j+1}(I))$ by equation (2.1), as desired.

Case 2. Assume that for all $h \in \Omega$ we have $h < \ell$. Choose some $h \in \Omega$. Then, $u >_{\mathrm{lex}} x_\ell(u/x_h) \in G(I)$. Set $w = x_\ell(u/x_h)$. Hence, this time $h \in \mathrm{set}(w)$. Note that $h \in \mathrm{supp}(u)$ and since $w_1 = \mathbf{x}_F u$ is squarefree, $h \notin F$. We are going to show that $F \subseteq \mathrm{set}(w)$. Hence, we will have $F \cup \{h\} \subseteq \mathrm{set}(w)$ and then the desired conclusion:

$$\mathrm{lcm}(w_1, w_2) = x_\ell \mathbf{x}_F u = x_h \mathbf{x}_F (x_\ell(u/x_h)) = \mathbf{x}_{F\cup\{h\}} w \in G(\mathrm{HS}_{j+1}(I)).$$

Let $m \in F$, then for some $p \neq m$, $x_m(u/x_p) \in G(I)$ and $x_m(u/x_p) >_{\mathrm{lex}} u$. So $m < p$.

Subcase 2.1. Let $p = h$. Then $\ell > h = p > m$. Hence $x_m(u/x_p) = x_m(u/x_h) >_{\mathrm{lex}} x_\ell(u/x_h) = w$. Whence, $m \in \mathrm{set}(w)$ in this case, as desired.

Subcase 2.2. Let $p \neq h$. Then $d(x_m(u/x_p), w) = d(x_m(u/x_p), x_\ell(u/x_h)) = 2$, $h \neq m$, $h \neq \ell$, $\ell \neq m$, $p \neq h$, $p \neq m$, and

$$\begin{aligned} &\deg_{x_h}(w) < \deg_{x_h}(x_m(u/x_p)), &\qquad &\deg_{x_\ell}(w) > \deg_{x_\ell}(x_m(u/x_p)),\\ &\deg_{x_m}(w) < \deg_{x_m}(x_m(u/x_p)), &\qquad &\deg_{x_p}(w) > \deg_{x_p}(x_m(u/x_p)). \end{aligned}$$

Whereas, for all $q \neq h, \ell, m, p$ we have $\deg_{x_q}(w) = \deg_{x_q}(x_m(u/x_p))$. Since $x_m(u/x_p)$, $w \in G(I)$ and $\deg_{x_m}(x_m(u/x_p)) > \deg_{x_m}(w)$, by the dual exchange property $(**)$ we have either $x_m(w/x_\ell) \in G(I)$ or $x_m(w/x_p) \in G(I)$.

Subcase 2.2.1. Assume $x_m(w/x_\ell) \in G(I)$. Note that

$$x_m(w/x_\ell) = x_m(x_\ell(u/x_h))/x_\ell = x_m(u/x_h) \in G(I).$$

Thus $m \in \Omega$ and by assumption $m < \ell$. Hence $x_m(w/x_\ell) >_{\text{lex}} w$ and so $m \in \text{set}(w)$.

SUBCASE 2.2.2. Assume $x_m(w/x_p) \in G(I)$. In this case, since $m < p$ we have $x_m(w/x_p) >_{\text{lex}} w$, and again $m \in \text{set}(w)$. The proof is complete □

Proposition 4 and Theorem 1 yield another proof of Conjecture 1 for matroidal ideals, one was already obtained in [3, Theorem 2.2].

Corollary 2. [3, Theorem 2.2] *Let $I \subset S$ be a matroidal ideal. Then $\text{HS}_j(I)$ is again a matroidal ideal for all j.*

Unfortunately, in general we could have $\text{HS}_{j+1}(I) \neq (\text{HS}_1(\text{HS}_j(I)))_{>j+1}$.

Example 1. Let $I = (x_1, x_2, x_3, x_4)(x_3, x_4, x_5) \subset S = K[x_1, \dots, x_5]$. Using the *Macaulay2* [10] package `HomologicalShiftIdeals` [8] we verified that $\text{HS}_j(I)$ is polymatroidal for all j. We have $\text{HS}_{j+1}(J) \neq (\text{HS}_1(\text{HS}_j(J)))_{>j+1}$ for $j = 1, 2, 3, 4$. For instance

$$(x_1x_2x_3x_4x_5)x_3^2x_4^2 \in G(\text{HS}_1(\text{HS}_1(J))) \setminus G(\text{HS}_2(J)).$$

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