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Well posedness and exponential stability of a thermoelastic Shear beam model

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Abstract

In this work, we investigate a thermoelastic shear beam model with thermal dissipation. We prove a well-posedness result using the Faedo–Galerkin method and establish exponential stability through the multiplier method. Our results improve the stability findings for certain thermoelastic Timoshenko-type systems as we do not require any relationship between the wave speeds, since our system has only one wave speed. Finally, we present some numerical experiments to illustrate our theoretical findings.

Key Words: Shear beam model, well–posedness, multiplier method, exponential stability, Faedo–Galerkin method.
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1 Introduction

This work is devoted to the study of the well-posedness and the decay rate of a shear beam model coupled with the heat equation governed by Fourier's law of heat conduction. The shear model is an improvement over the Euler-Bernoulli beam model by incorporating the shear distortion effect but without rotary inertia.

Historically, the classical Euler-Bernoulli differential equation for vibrations of a uniform beam is given by

$$\rho A\varphi_{tt} + EI\varphi_{xxxx} = 0,$$

where ρ represents the mass density, A denotes the cross-sectional area, E stands for the modulus of elasticity, I is the inertia momentum of the cross-sectional area of the beam, and $\varphi(x,t)$ represents the deflection of the beam at the point $x \in (0,L)$ and time $t > 0$. The Euler-Bernoulli model does not account for either rotatory inertia or shear deformation. Accordingly, the Euler-Bernoulli model slightly overestimates the natural frequencies. Rayleigh [21] incorporated the rotational movements of the cross section. From a mathematical modeling perspective, the angle of rotation is equal to the slope of the deflection curve φ_x. Consequently, the differential equation for the Rayleigh model is given by

$$\rho A\varphi_{tt} + EI\varphi_{xxxx} - \rho I\varphi_{xxtt} = 0.$$

As a result, it partially corrects the overestimation of natural frequencies in the Euler-Bernoulli model. However, the natural frequencies are still being overestimated.

The addition of a shear distortion to the Euler-Bernoulli model is known as the shear model. In contrast to the Euler-Bernoulli and the Rayleigh beam models, the shear model has two dependent variables, and the total angle of rotation is the difference between β, which represents the angle of distortion due to shear and ψ, which represents the angle of rotation of the cross-section due to the bending moment. The total angle is approximately equal to the derivative of the deflection angle φ, and it is expressed as $\varphi_x = \beta - \psi$. These two dependent variables are represented by the coupled equations

$$\begin{cases} \rho\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x = 0, & \text{in } (0, L) \times (0, +\infty), \\ -b\psi_{xx} - \kappa(\varphi_x + \psi) = 0, & \text{in } (0, L) \times (0, +\infty). \end{cases}$$

The adding of shear distortion to the Euler-Bernoulli beam improves considerably the estimation of the natural frequencies.

The most significant improvement to the Euler-Bernoulli model was made by Timoshenko in 1921. In his paper [23], he proposed a beam theory that incorporates the effects of shear deformation and rotational inertia into the Euler-Bernoulli beam theory. The equations of motion are given by

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0 & \text{in } (0, L) \times (0, \infty), \\ \rho_2\psi_{tt} - \psi_{xx} + \kappa(\varphi_x + \psi) = 0 & \text{in } (0, L) \times (0, \infty), \end{cases} \tag{1}$$

where $\rho_1 = \rho A, \rho_2 = \rho I, \kappa = k'GA, b = EI$, k' is the transverse shear factor and G is the shear modulus.

The longtime behavior of beam models subjected to different dissipation mechanisms has been extensively studied in the last four decades. Chen *et al.* [7] controlled the Euler-Bernoulli equation by a locally distributed viscous damping and proved that the energy of the beam decays exponentially. Chen and collaborators [5, 6, 7] considered boundary dissipations acting on one end of the beam, and proved the exponential decay of the energy. Liu and Liu [9] studied the longitudinal and the transversal vibrations of Euler-Bernoulli beam with locally distributed Kelvin-Voigt damping. They established an exponential decay for the transversal vibration and non exponential decay for the longitudinal vibrations.

Rao [19] investigated a Rayleigh beam clamped at one end with boundary control applied at the other end. He established an exponential stability in the case of moment control. Additionally, he proved strong asymptotic stabilization and the absence of exponential stabilization when boundary viscous damping is applied. In [20], he further discussed the Rayleigh beam clamped at both ends with a viscous damping in the feedback and obtained an optimal exponential decay rate.

Great importance has been given to the study of the longtime behavior of Timoshenko systems. Thousands of papers have appeared during the last three decades whose subject is the study of effect of various dissipation mechanisms. In a pioneer work, Soufyane [22] examined the impact of a localized elastic damping $b(x)\psi_t$ acting only on the equation of rotational angle

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x = 0, & (x,t) \in (0, L) \times (0, +\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) + b(x)\psi_t = 0, & (x,t) \in (0, L) \times (0, +\infty), \end{cases}$$

where, b is a positive continuous function. He showed that the exponential decay occurs if and only if, the wave propagation velocities are equal

$$\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}. \tag{2}$$

This result was extended by Muñoz Rivera and Racke [17] to the case of indefinite damping $b(x)$ satisfying $\|b - \bar{b}\|_{L^2} < \varepsilon$ for a small $\varepsilon > 0$, where

$$\bar{b} = \frac{1}{L}\int_0^L b(x)dx > 0.$$

Other dissipation mechanisms have considered subsequently. Viscoelastic damping of memory type was examined by Ammar Khodja *et al.* [4]. Muñuz Rivera and Fernández Sare [14] treated Timoshenko systems with infinite history. In both cases the exponential rate of decay was obtained under suitable conditions on the relaxation functions and provided that (2) holds. Messaoudi and Said Houari [13] improved the result of [14] by assuming weaker conditions on the relaxation function. Muñoz Rivera and Racke [15] considered a thermal dissipation induced through coupling with the heat equation. They achieved an exponential rate of decay provided that (2) holds. The same result was obtained by Messaoudi and Said Houari [12] for thermal damping produced by heat conduction governed by the Green and Naghedi theory of type III.

Note that the equality of wave speeds (2) is purely mathematical. Indeed, if (2) holds, then

$$\frac{k'G}{\rho} = \frac{E}{\rho},$$

which leads to $G = \dfrac{E}{k'}$, and since the relation between tensions G and E is given by

$$G = \frac{E}{2+2\nu},$$

where the Poisson ratio ν satisfies $0 < \nu < 1/2$, we arrive at

$$\frac{k'}{2+2\nu} = 1$$

and this is not true because the shear factor k' is always a small number less than one.

If (2) does not hold, Timoshenko-type systems with a single dissipation mechanism lose the exponential decay; in most cases, polynomial decay rates can be established. See, e.g., [2, 16, 15, 11, 14, 10].

To the best of our knowledge, the stability of the shear beam model was only investigated by Almeida Jùnior *et al.* [3], Ramos *et al.* [18], and recently, by Ahmima *et al.* [1]. In [3] the authors considered the following system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x + \mu\varphi_t = 0 & \text{in } (0, L) \times (0, \infty), \\ -b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) = 0 & \text{in } (0, L) \times (0, \infty), \end{cases}$$

where the frictional damping $\mu\varphi_t$ acts on the transversal displacement equation. They proved the exponential decay of the energy regardless to any relationship between the coefficients of the system. Whereas the authors of [18] examined the effect of a feedback damping acting on the rotational angle and showed that the system is non-exponentially stable. In [1], the authors studied a shear beam system with type III thermoelasticity and established an exponential decay rate.

In this paper, we are concerned with the following shear model system with thermal dissipation governed by the heat equation based on the classical law of thermal conduction,

$$\begin{cases} \rho\varphi_{tt} - \kappa(\varphi_x + \psi)_x + \mu\theta_x = 0 & \text{in } (0, L) \times (0, \infty)\,, \\ -b\psi_{xx} + \kappa(\varphi_x + \psi) = 0 & \text{in } (0, L) \times (0, \infty)\,, \\ c\theta_t - \delta\theta_{xx} + \mu\varphi_{xt} = 0 & \text{in } (0, L) \times (0, \infty)\,, \end{cases} \tag{3}$$

where θ is the difference of the temperature from the configuration value T_0, and $c > 0$ is a physical constant that characterizes the heat conductivity of the material. In addition, we equip system (3) with the following initial and boundary conditions

$$\varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x),\ \psi(x, 0) = \psi_0(x),\ \theta(x, 0) = \theta_0(x),\ x \in [0, L] \tag{4}$$

$$\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = \theta_x(0, t) = \theta_x(L, t) = 0,\ t \in (0, +\infty)\,. \tag{5}$$

In order to investigate the dissipative nature of the problem (3)–(5), we define the energy functional as:

$$E(t) := \frac{\rho}{2}\int_0^L \varphi_t^2 dx + \frac{b}{2}\int_0^L \psi_x^2 dx + \frac{c}{2}\int_0^L \theta^2 dx + \frac{\kappa}{2}\int_0^L (\varphi_x + \psi)^2\, dx. \tag{6}$$

We have the following result

Lemma 1.1. *The energy functional satisfies, along any weak solution (φ, ψ, θ) of (3)–(5), the estimate*

$$\frac{d}{dt}E(t) = -\delta\int_0^L \theta_x^2 dx \le 0. \tag{7}$$

Proof. Performing the L^2-inner product of the equations of (3) by $\varphi_t, \psi_t, \theta$, respectively, we get

$$\frac{\rho}{2}\frac{d}{dt}\int_0^L \varphi_t^2 dx + \kappa\int_0^L (\varphi_x + \psi)\varphi_{xt} dx - \mu\int_0^L \theta\varphi_{xt} dx = 0,$$

$$\frac{b}{2}\frac{d}{dt}\int_0^L \psi_x^2 dx + \kappa\int_0^L (\varphi_x + \psi)\psi_t dx = 0,$$

$$\frac{c}{2}\frac{d}{dt}\int_0^L \theta^2 dx + \mu\int_0^L \theta\varphi_{xt} dx = -\delta\int_0^L \theta_x^2 dx.$$

By adding of the above equations, estimate (7) follows immediately for any regular solution. The validity for weak solutions is established through density arguments. ∎

Note that since Neumann boundary conditions are assumed for θ, we are unable to apply Poincaré's inequality for θ. To address this challenge, we proceed as follows:
From the third equation of (3) and boundary conditions (4), we have

$$\frac{d}{dt}\int_0^L \theta(x, t) dx = 0.$$

Thus,

$$\int_0^L \theta(x, t) dx = \int_0^L \theta_0(x) dx.$$

So, if we set

$$\widetilde{\theta}(x,t) = \theta(x,t) - \frac{1}{L}\int_0^L \theta_0(x)dx,$$

we get $\left(\varphi, \psi, \widetilde{\theta}\right)$ satisfies the problem (3)–(5), with initial condition for $\widetilde{\theta}$ given by

$$\widetilde{\theta}(x,0) = \theta_0(x) - \frac{1}{L}\int_0^L \theta_0(x)dx$$

and more importantly, we have

$$\int_0^L \widetilde{\theta}(x,t)dx = 0, \tag{8}$$

which allows the application of Poincaré's inequality for $\widetilde{\theta}$. In the sequel, we work with $\left(\varphi, \psi, \widetilde{\theta}\right)$ but, for convenience, we write (φ, ψ, θ).

2 Well-posedness

In this section, we use the Faedo–Galerkin method to prove the existence of a weak solution to the problem (3)–(5). First, we introduce the phase space:

$$\mathcal{H} := H_0^1(0,L) \times H_0^1(0,L) \times H_*^1(0,L) \tag{9}$$

where

$$H_*^1(0,L) := \left\{ \phi \in H^1(0,L),\ \int_0^L \phi(x)dx = 0 \right\}. \tag{10}$$

Note that $H_*^1(0,L)$ is a closed subspace of $H^1(0,L)$. Therefore, it is also a Hilbert space, and so is $\mathcal{H}$.

Definition 2.1. *Let* $(\varphi_0, \varphi_1, \psi_0, \theta_0) \in H_0^1(0,L) \times\ L^2(0,L) \times\ H_0^1(0,L) \times H_*^1(0,L)$. *A weak solution of* (3)–(5) *is a function* $U = (\varphi, \psi, \theta) \in L^\infty(0,T;\mathcal{H})$, *such that*

$$\left\{\begin{aligned} \rho\frac{d}{dt}\int_0^L \varphi_t u dx + \kappa\int_0^L (\varphi_x + \psi)\, u_x dx - \mu\int_0^L \theta u_x dx &= 0 \quad \forall u \in H_0^1(0,L),\\ b\int_0^L \psi_x u_x dx + \kappa\int_0^L (\varphi_x + \psi)\, u dx &= 0 \quad \forall u \in H_0^1(0,L),\\ c\frac{d}{dt}\int_0^L \theta\nu dx + \delta\int_0^L \theta_x \nu_x dx - \mu\int_0^L \varphi_t \nu_x dx &= 0 \quad \forall \nu \in H_*^1(0,L), \end{aligned}\right. \tag{11}$$

for almost every $t \in [0,T]$, *and*

$$\varphi(0) = \varphi_0,\ \varphi_t(0) = \varphi_1,\ \psi(0) = \psi_0,\ \theta(0) = \theta_0.$$

The following lemma is useful in the proof of our well-posedness result.

Lemma 2.2. *[8] Let $V \subset X \subset W$ be Banach spaces such that V, W are reflexive and the embedding $V \subset W$ is compact. If $\{w^n\}$ is bounded sequence in $L^p(0,T;V)$ and $\{w_t^n\}$ is bounded in $L^q(0,T;W)$, $1 < p, q < +\infty$, then, there exists a subsequence $\{w^k\}$ such that*

$$w^k \longrightarrow w \text{ in } L^p(0,T;X).$$

Our well-posedness result reads as follows:

Theorem 2.3. *For any initial data $(\varphi_0, \varphi_1, \psi_0, \theta_0) \in H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times H_*^1(0,L)$ and any $T > 0$, the problem (3)–(5) has a weak solution (φ, ψ, θ) such that*

$$\begin{gathered}\varphi \in L^\infty\left(0,T;H_0^1(0,L)\right),\ \varphi_t \in L^\infty\left(0,T;L^2(0,L)\right),\\ \psi \in L^\infty\left(0,T;H_0^1(0,L)\right),\ \theta \in L^\infty\left(0,T;L^2(0,L)\right) \cap L^2(0,T;H_*^1(0,L)).\end{gathered}$$

Proof. The proof will be carried out using the Faedo–Galerkin method through the following steps:

Step 1: Approximate problem

Let $\{\omega_j\}_{j=1}^\infty$ and $\{\phi_j\}_{j=1}^\infty$ be orthonormal bases in $H_0^1(0,L)$ and $H_*^1(0,L)$ respectively, formed by the eigenfunctions of the operator $-\dfrac{\partial^2}{\partial x^2}$, associated with the eigenvalues $\{\lambda_j\}$ and $\{\mu_j\}$ respectively. We consider the finite dimensional spaces

$$H_n = span\{\omega_1, \omega_2, \cdots, \omega_n\}, \quad V_n = span\{\phi_1, \phi_2, \cdots, \phi_n\},$$

and set

$$\mathcal{H}_n = H_n \times H_n \times V_n.$$

Let

$$\begin{aligned}\varphi_0^n &:= \sum_{j=1}^n \langle \varphi_0, \omega_j \rangle \omega_j \longrightarrow \varphi_0 \text{ in } H_0^1(0,L),\\ \varphi_1^n &:= \sum_{j=1}^n \langle \varphi_1, \omega_j \rangle \omega_j \longrightarrow \varphi_1 \text{ in } L^2(0,L),\\ \psi_0^n &:= \sum_{j=1}^n \langle \psi_0, \omega_j \rangle \omega_j \longrightarrow \psi_0 \text{ in } H_0^1(0,L),\\ \theta_0^n &:= \sum_{j=1}^n \langle \theta_0, \phi_j \rangle \phi_j \longrightarrow \theta_0 \text{ in } H_*^1(0,L).\end{aligned} \tag{12}$$

For any $n \in \mathbb{N}$, we look for a solution $(\varphi^n, \psi^n, \theta^n) \in L^\infty(0,T;\mathcal{H}_n)$, in the form

$$\varphi^n(x,t) = \sum_{j=1}^n a_{j,n}(t)\omega_j(x),\ \psi^n(x,t) = \sum_{j=1}^n b_{j,n}(t)\omega_j(x),\ \theta^n(x,t) = \sum_{j=1}^n c_{j,n}(t)\phi_j(x),$$

that satisfy the following approximate problem

$$\begin{cases} \rho \int_0^L \varphi_{tt}^n \omega_j dx + \kappa \int_0^L (\varphi_x^n + \psi^n) \omega_{jx} dx - \mu \int_0^L \theta^n \omega_{jx} dx = 0, \\ b \int_0^L \psi_x^n \omega_{jx} dx + \kappa \int_0^L (\varphi_x^n + \psi^n) \omega_j dx = 0, \\ c \int_0^L \theta_t^n \phi_j dx + \delta \int_0^L \theta_x^n \phi_{jx} dx - \mu \int_0^L \varphi_t^n \phi_{jx} dx = 0, \end{cases} \tag{13}$$

and the initial conditions

$$(\varphi^n(0), \varphi_t^n(0), \psi^n(0), \theta^n(0)) = (\varphi_0^n, \varphi_1^n, \psi_0^n, \theta_0^n). \tag{14}$$

The insertion of $\varphi^n, \psi^n, \theta^n$ into (13) leads to the following system of linear ordinary differential equations

$$\begin{cases} \rho_1 a_{j,n}'' + \lambda_j \kappa a_{j,n} + \kappa \sum_{k=1}^n \langle \omega_{kx}, \omega_j \rangle b_{k,n} + \mu \sum_{k=1}^n \langle \phi_{kx}, \omega_j \rangle c_{k,n} = 0, \\ (b\lambda_j + \kappa) b_{j,n} + \kappa \sum_{k=1}^n \langle \omega_{kx}, \omega_j \rangle a_{k,n} = 0, \quad 1 \le j \le n. \\ c c_{j,n}' + \delta \mu_j c_{j,n} + \mu \sum_{k=1}^n \langle \omega_{kx}, \phi_j \rangle a_{k,n}' = 0, \end{cases} \tag{15}$$

Solving the second equation of (15) for $b_{j,n}$, $1 \le j \le n$, and then inserting the obtained solutions in the first equation of (15), we arrive at a system of $2n-$ordinary differential equations for $2n$ unknown functions $a_{j,n}, c_{j,n},\ j = 1, \cdots, n$, which satisfy the initial conditions

$$a_{j,n}(0) = \langle \varphi_0^n, \omega_j \rangle, \ a_{j,n}'(0) = \langle \varphi_1^n, \omega_j \rangle, \ c_{j,n}(0) = \langle \theta_0^n, \phi_j \rangle, \quad j = 1, ..., n. \tag{16}$$

The standard theory of ordinary differential equations guarantees the existence of functions $a_{j,n} \in C^2([0, t_n])$ and $c_{j,n} \in C^1([0, t_n])$ that solve (15)–(16) for $0 < t_n \le T$. Then, from the second equation of (15), the existence of $b_{j,n} \in C^2([0, t_n])$ is established. This completes the proof of the existence of a unique local solution $U_n = (\varphi^n, \psi^n, \theta^n)$ to the approximate problem (13)–(14) on a maximal interval $[0, t_n)$, with $0 \le t_n \le T$, for all $n \in \mathbb{N}$.

Step 2. Energy estimates for approximate solutions

Multiplying the equations of (13), respectively, by $a_{j,n}', b_{j,n}'$ and $c_{j,n}'$, then adding the resulting equations over $j = 1, 2, \cdots, n$, we obtain

$$\begin{cases} \rho \int_0^L \varphi_{tt}^n \varphi_t^n dx + \kappa \int_0^L (\varphi_x^n + \psi^n) \varphi_{xt}^n dx - \mu \int_0^L \theta^n \varphi_{xt}^n dx = 0, \\ b \int_0^L \psi_x^n \psi_{xt}^n dx + \kappa \int_0^L (\varphi_x^n + \psi^n) \psi_t^n dx = 0, \\ c \int_0^L \theta_t^n \theta^n dx + \kappa \int_0^L \theta_x^n \theta_x^n dx + \mu \int_0^L \varphi_{xt}^n \theta^n dx = 0, \end{cases}$$

which implies that

$$\frac{d}{dt}\left[\frac{\rho}{2}\int_0^L (\varphi_t^n)^2\,dx + \frac{b}{2}\int_0^L (\psi_x^n)^2 dx + \frac{\kappa}{2}\int_0^L (\varphi_x^n+\psi^n)^2\,dx + \frac{c}{2}\int_0^L (\theta^n)^2 dx\right] + \delta\int_0^L (\theta_x^n)^2 dx = 0. \tag{17}$$

An integration over $[0,t]$ for $0<t\le t_n$ gives

$$E_n(t) + \delta\int_0^t\int_0^L (\theta_x^n)^2 dxds \le C_1, \tag{18}$$

where

$$E_n(t) := \frac{\rho}{2}\int_0^L (\varphi_t^n)^2 dx + \frac{b}{2}\int_0^L (\psi_x^n)^2 dx + \frac{\kappa}{2}\int_0^L (\varphi_x^n+\psi^n)^2\,dx + \frac{c}{2}\int_0^L (\theta^n)^2 dx$$

and C_1 is a positive constant depending only on the initial data. Consequently, t_n is independent of n and the solution of the problem (13) can be extended to the whole interval $[0,T]$.

Next, taking the supremum of (18) over $[0,T]$, we infer that, for every $n\in\mathbb{N}$, we have

$$\sup_{t\in[0,T]} E_n(t) + \delta \sup_{t\in[0,T]}\int_0^t\int_0^L (\theta_x^n(s))^2 dxds \le C_1;$$

that is,

$$\begin{aligned}&\|\varphi^n\|_{L^\infty(0,T;H_0^1(0,L))} + \|\varphi_t^n\|_{L^\infty(0,T;L^2(0,L))} + \|\psi^n\|_{L^\infty(0,T;H_0^1(0,L))}\\ &\quad + \|\theta^n\|_{L^\infty(0,T;L_*^2(0,L))} + \delta\|\theta^n\|^2_{L^2(0,T;H_*^1(0,L))} \le C_1.\end{aligned} \tag{19}$$

Step 3. Convergence of the approximate solutions

From (19), we deduce that

$$\begin{array}{l} \{\varphi^n\} \text{ is bounded in } L^\infty(0,T;H_0^1(0,L)),\\ \{\varphi_t^n\} \text{ is bounded in } L^\infty(0,T;L^2(0,L)),\\ \{\psi^n\} \text{ is bounded in } L^\infty(0,T;H_0^1(0,L)),\\ \{\theta^n\} \text{ is bounded in } L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H_*^1(0,L)). \end{array} \tag{20}$$

Consequently, we can extract subsequences from $\{\varphi^n\}$, $\{\psi^n\}$, $\{\theta^n\}$, still denoted by $\{\varphi^n\}$, $\{\psi^n\}$, $\{\theta^n\}$, such that

$$\begin{array}{c} \varphi^n \overset{*}{\rightharpoonup} \varphi \text{ in } L^\infty(0,T;H_0^1(0,L)) \text{ and } \varphi^n \rightharpoonup \varphi \text{ in } L^2(0,T;H_0^1(0,L)),\\ \varphi_t^n \overset{*}{\rightharpoonup} \varphi_t \text{ in } L^\infty(0,T;L^2(0,L)), \text{ and } \varphi_t^n \rightharpoonup \varphi_t \text{ in } L^2(0,T;L^2(0,L)),\\ \psi^n \overset{*}{\rightharpoonup} \psi \text{ in } L^\infty(0,T;H_0^1(0,L)), \text{ and } \psi^n \rightharpoonup \psi \text{ in } L^2(0,T;H_0^1(0,L)),\\ \theta^n \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(0,T;L_*^2(0,L)), \text{ and } \theta^n \rightharpoonup \theta \text{ in } L^2(0,T;H_*^1(0,L)). \end{array} \tag{21}$$

From (21) and Lemma 2.2, we infer that there exists a subsequence $\left\{\varphi^k\right\}$ such that

$$\varphi^k \longrightarrow \varphi \text{ strongly in } L^2\left(0,T;L^2(0,L)\right) \text{ and } \varphi^k \longrightarrow \varphi \,\text{a.e. in } (0,L)\times(0,T)\,. \tag{22}$$

Rewriting (13) with k instead of n and integrating with respect to t over $[0,t]$ for $0 \leq t \leq T$, we arrive at

$$\left\{ \begin{aligned} \rho \int_0^t \int_0^L \varphi_{tt}^k \omega_j dxds + \kappa \int_0^t \int_0^L \left(\varphi_x^k + \psi^k\right) \omega_{jx} dxds - \mu \int_0^t \int_0^L \theta^k \omega_{jx} dxds = 0, \\ b \int_0^t \int_0^L \psi_x^k \omega_{jx} dxds + \kappa \int_0^t \int_0^L \left(\varphi_x^k + \psi^k\right) \omega_j dxds = 0, \\ c \int_0^t \int_0^L \theta_t^k \phi_j dxds + \delta \int_0^t \int_0^L \theta_x^k \phi_{jx} dxds - \mu \int_0^t \int_0^L \varphi_t^k \phi_{jx} dxds = 0. \end{aligned} \right. \tag{23}$$

Using (21), we get

$$\int_0^t \int_0^L \left(\varphi_x^k + \psi^k\right) \omega_{jx} dxds \to \int_0^t \int_0^L \left(\varphi_x + \psi\right) \omega_{jx} dxds, \tag{24}$$

$$\int_0^t \int_0^L \theta^k \omega_{jx} dxds \to \int_0^t \int_0^L \theta \omega_{jx} dxds, \tag{25}$$

$$\int_0^t \int_0^L \psi_x^k \omega_{jx} dxds \to \int_0^t \int_0^L \psi_x \omega_{jx} dxds, \tag{26}$$

$$\int_0^t \int_0^L \left(\varphi_x^k + \psi^k\right) \omega_j dxds \to \int_0^t \int_0^L \left(\varphi_x + \psi\right) \omega_j dxds, \tag{27}$$

$$\int_0^t \int_0^L \theta_x^k \phi_{jx} dxds \to \int_0^t \int_0^L \theta_x \phi_{jx} dxds, \tag{28}$$

$$\int_0^t \int_0^L \varphi_t^k \phi_{jx} dxds \to \int_0^t \int_0^L \varphi_t \phi_{jx} dxds \tag{29}$$

and

$$\int_0^t \int_0^L \varphi_t^k \omega_j dxds \to \int_0^t \int_0^L \varphi_t \omega_j dxds. \tag{30}$$

Rewriting the first equation of (23) replacing ω_j with $\phi \in H_0^1(0,L)$, applying Young's inequality and taking the supremum with respect to $\|\phi\|_{H_0^1} = 1$, we obtain

$$\rho \int_0^T \|\varphi_{tt}^k\|_{H^{-1}}^2 \, dt \leq \int_0^T \left(\kappa \|\varphi_x^k + \psi^k\|^2 + \mu \|\theta^k\|\right) dt \leq C,$$

which yields

$$\|\varphi_{tt}^k\|_{H^{-1}} \leq C, \quad \text{for a.e. } t \in (0,T),$$

and the latter implies that: $\left(\varphi_{tt}^k\right)$ is bounded in $L^2(0,T;H^{-1}(0,L))$.

From Lemma 2.2 and (21) we infer that, for a subsequence $\{\varphi^k\}$, we have

$$\varphi_t^k \to \varphi_t \text{ strongly in } L^2(0,T;H^{-1}(0,L)) \text{ and } \varphi_t^k \to \varphi_t \text{ a.e. in } (0,T) \times (0,L).$$

Therefore

$$\int_0^t \int_0^L \varphi_{tt}^k \omega_j dxds = \frac{d}{dt} \int_0^t \int_0^L \varphi_t^k \omega_j dxds \to \int_0^t \int_0^L \varphi_{tt} \omega_j dxds, \quad \forall \omega_j \in L^2(0,L). \tag{31}$$

Similarly,

$$\int_0^t \int_0^L \theta_t^k \phi_{jx} dx ds \to \int_0^t \int_0^L \theta_t \phi_{jx} dx ds, \quad \forall \phi_j \in H_*^1(0,L). \tag{32}$$

From (24)–(32), we deduce that

$$\begin{cases} \rho \int_0^t \int_0^L \varphi_{tt} \omega_j dx ds + \kappa \int_0^t \int_0^L (\varphi_x + \psi) \omega_{jx} dx ds - \mu \int_0^t \int_0^L \theta \omega_{jx} dx ds = 0, \\ b \int_0^t \int_0^L \psi_x \omega_{jx} dx ds + \kappa \int_0^t \int_0^L (\varphi_x + \psi) \omega_j dx ds = 0, \\ c \int_0^t \int_0^L \theta_t \phi_j dx ds + \delta \int_0^t \int_0^L \theta_x \phi_{jx} dx ds - \mu \int_0^t \int_0^L \varphi_t \phi_{jx} dx ds = 0. \end{cases} \tag{33}$$

The left hand sides of the system (33) are absolutely continuous as functions of t, and are therefore differentiable almost everywhere. By differentiating the equations (33) with respect to t, we arrive at

$$\begin{cases} \rho \int_0^L \varphi_{tt} \omega_j dx + \kappa \int_0^L (\varphi_x + \psi) \omega_{jx} dx - \mu \int_0^L \theta \omega_{jx} dx = 0, \\ b \int_0^L \psi_x \omega_{jx} dx + \kappa \int_0^L (\varphi_x + \psi) \omega_j dx = 0, \\ c \int_0^L \theta_t \phi_j dx + \delta \int_0^L \theta_x \phi_{jx} dx - \mu \int_0^L \varphi_t \phi_{jx} dx = 0, \end{cases} \tag{34}$$

for any $\omega_j, \phi_j,\ j \geq 1$. Since $\{\omega_j\}$, $\{\phi_j\}$ are dense in $H_0^1(0,L)$ and $H_*^1(0,L)$, respectively, (34) still valid with $u \in H_0^1(0,L), v \in H_*^1(0,L)$ instead of ω_j, ϕ_j, respectively. Therefore, (11) follows immediately, which shows that (φ, ψ, θ) is a weak solution of (3).

Step 4. Initial data

From Lemma 2.2 and (21), we infer that

$$\varphi^n \longrightarrow \varphi \text{ strongly in } C\left([0,T]; L^2(0,L)\right).$$

Therefore,

$$\varphi(0) = \varphi_0.$$

Next, integrating the third equation of (11) with respect to t over $(0,T)$, then taking $\nu = \eta(t)\phi(x)$ with $\eta \in C^\infty([0,T])$ such that $\eta(0) = 1$ and $\eta(T) = 0$, we get for all $\phi \in H_*^1(0,L)$

$$\begin{aligned} c \int_0^T \int_0^L \theta_t \eta(t) \phi(x) dx dt &+ \delta \int_0^T \int_0^L \theta_x \eta(t) \phi_x(x) dx dt \\ &+ \mu \int_0^T \int_0^L \varphi_t \eta(t) \phi_x(x) dx dt = 0. \end{aligned} \tag{35}$$

The integration by parts of the first term yields

$$\int_0^T \int_0^L \theta_t(x,t)\eta(t)\phi(x)dxdt = -\int_0^L \theta(x,0)\phi(x)dx \\ - \int_0^T \int_0^L \theta(x,t)\eta_t(t)\phi(x)dxdt.$$

The insertion in (35) leads to

$$c\int_0^L \theta(0)\phi(x)dx = \delta\int_0^T \int_0^L \theta_x\eta(t)\phi_x(x)dxdt + \mu\int_0^T \int_0^L \varphi_t\eta(t)\phi_x(x)dxdt \\ - c\int_0^T \int_0^L \theta(x,t)\eta_t(t)\phi(x)dxdt, \tag{36}$$

for all $\phi \in H^1_*(0,L)$.

Applying the same process to the third equation in (13), and recalling the initial conditions (14), we obtain

$$c\int_0^L \theta^n(0)\phi(x)dx = \delta\int_0^T \int_0^L \theta^n_x\eta(t)\phi_x(x)dxdt + \mu\int_0^T \int_0^L \varphi^n_t\eta(t)\phi_x(x)dxdt \\ - c\int_0^T \int_0^L \theta^n(x,t)\eta_t(t)\phi(x)dxdt. \tag{37}$$

Passing to the limit in (37), recalling (12) and (21), we obtain

$$c\int_0^L \theta_0\phi(x)dx = \delta\int_0^T \int_0^L \theta_x\eta(t)\phi_x(x)dxdt + \mu\int_0^T \int_0^L \varphi_t\eta(t)\phi_x(x)dxdt \\ - c\int_0^T \int_0^L \theta(x,t)\eta_t(t)\phi(x)dxdt, \tag{38}$$

for all $\phi \in H^1_*(0,L)$. The comparison of (36) and (38) leads to

$$\theta(0) = \theta_0.$$

Similarly, we get

$$\varphi_t(0) = \varphi_1 \text{ and } \psi(0) = \psi_0,$$

which completes the proof of the Theorem 2.3. □

3 Exponential decay

In this section, we state and prove the exponential stability of the energy associated to the solution of the problem (3)–(5).

Our main result reads as follows:

Theorem 3.1. *There exist two positive constants A, $\gamma > 0$, such that the energy $E(t)$, defined by (6), satisfies, along the solution (φ, ψ, θ), the estimate*

$$E(t) \leq E(0)Ae^{-\gamma t}, \ \forall t \geq 0. \tag{39}$$

Proof. The proof of Theorem 3.1 will be performed through several lemmas. ∎

Lemma 3.2. *Let (φ, ψ, θ) be a weak solution of (3)–(5), then the functional*

$$\mathcal{F}_1(t) := c\int_0^L \varphi_t \int_0^x \theta(y)dydx \tag{40}$$

satisfies, for any $\varepsilon > 0$, the estimate

$$\begin{aligned}\mathcal{F}_1'(t) \leq& -\frac{\mu}{2}\int_0^L \varphi_t^2 dx + \varepsilon\int_0^L (\varphi_x + \psi)^2 dx \\ &+ \left(\frac{\delta^2}{2\mu} + \frac{c\mu c_P}{\rho} + \frac{c^2\kappa^2 c_P}{4\varepsilon\rho^2}\right)\int_0^L \theta_x^2 dx,\end{aligned} \tag{41}$$

where, $c_P > 0$ is the Poincaré constant.

Proof. Differentiating $\mathcal{F}_1(t)$, using equations (3), we get

$$\begin{aligned}\mathcal{F}_1'(t) =& \frac{c}{\rho}\int_0^L (\kappa(\varphi_x + \psi)_x - \mu\theta_x)\int_0^x \theta(y)dydx \\ &+ \int_0^L \varphi_t \int_0^x (\delta\theta_{xx}(y) - \mu\varphi_{xt}(y))\, dydx.\end{aligned}$$

An integration by parts, using (8) and Young's inequality, yields

$$\begin{aligned}\mathcal{F}_1'(t) =& -\frac{c}{\rho}\int_0^L (\kappa(\varphi_x + \psi) - \mu\theta)\,\theta dx + \int_0^L \varphi_t\,(\delta\theta_x - \mu\varphi_t)\, dx, \\ =& -\mu\int_0^L \varphi_t^2 dx + \frac{c\mu}{\rho}\int_0^L \theta^2 dx + \delta\int_0^L \varphi_t\theta_x dx - \frac{c\kappa}{\rho}\int_0^L (\varphi_x + \psi)\,\theta dx, \\ \leq& -\frac{\mu}{2}\int_0^L \varphi_t^2 dx + \frac{\delta^2}{2\mu}\int_0^L \theta_x^2 dx + \varepsilon\int_0^L (\varphi_x + \psi)^2 dx \\ &+ \left(\frac{c\mu}{\rho} + \frac{c^2\kappa^2}{4\varepsilon\rho^2}\right)\int_0^L \theta^2 dx.\end{aligned}$$

Then Poincaré's inequality leads to the estimate (41). ∎

Lemma 3.3. *Let (φ, ψ, θ) be a weak solution of (3)–(5), then the functional*

$$\mathcal{F}_2(t) := \rho\int_0^L \varphi_t\varphi dx \tag{42}$$

satisfies the estimate

$$\mathcal{F}_2'(t) \leq -\frac{b}{2}\int_0^L \psi_x^2 dx - \frac{\kappa}{2}\int_0^L (\varphi_x+\psi)^2 dx + \rho\int_0^L \varphi_t^2 dx + M\int_0^L \theta^2 dx, \tag{43}$$

for a positive constant $M > 0$.

Proof. Differentiating $\mathcal{F}_2(t)$, using equations (3), the boundary conditions (4) and integration by parts, we get

$$\begin{aligned}\mathcal{F}_2'(t) =&\rho\int_0^L \varphi_t^2 dx - \kappa\int_0^L (\varphi_x+\psi)\varphi_x dx - \mu\int_0^L \theta\varphi_x dx,\\ =&\rho\int_0^L \varphi_t^2 dx - \kappa\int_0^L (\varphi_x+\psi)^2 dx + \kappa\int_0^L (\varphi_x+\psi)\psi dx - \mu\int_0^L \theta\varphi_x dx.\end{aligned} \tag{44}$$

From the second equation of (3) we have

$$\kappa\int_0^L (\varphi_x+\psi)\psi dx = -b\int_0^L \psi_x^2 dx. \tag{45}$$

Inserting (45) into (44) and using Young's inequality, we infer that for any $\varepsilon_1 > 0$,

$$\mathcal{F}_2'(t) \leq -\kappa\int_0^L (\varphi_x+\psi)^2 dx - b\int_0^L \psi_x^2 dx + \rho\int_0^L \varphi_t^2 dx + \varepsilon_1\int_0^L \varphi_x^2 dx + \frac{\mu^2}{4\varepsilon_1}\int_0^L \theta^2 dx.$$

The fact that $\varphi_x^2 \leq 2(\varphi_x+\psi)^2 + 2\psi^2$ and Poincaré's inequality lead to

$$\begin{aligned}\mathcal{F}_2'(t) \quad &\leq -(b-2\varepsilon_1 c_P)\int_0^L \psi_x^2 dx - (\kappa-2\varepsilon_1)\int_0^L (\varphi_x+\psi)^2 dx\\ &+\rho\int_0^L \varphi_t^2 dx + \frac{\mu^2 c_P}{4\varepsilon_3}\int_0^L \theta_x^2 dx.\end{aligned}$$

Choosing $\varepsilon_1 \leq \min\left((\frac{\kappa}{4}, \frac{b}{4c_P}\right)$, we get

$$\mathcal{F}_2'(t) \leq -\frac{b}{2}\int_0^L \psi_x^2 dx - \frac{\kappa}{2}\int_0^L (\varphi_x+\psi)^2 dx + \rho\int_0^L \varphi_t^2 dx + M\int_0^L \theta_x^2 dx,$$

for some positive constant M.

□

Now we are in the position to prove Theorem 3.1. In order to do that, let us define the Lyapunov functional $\mathcal{L}$ by

$$\mathcal{L}(t) := NE(t) + N_1\mathcal{F}_1(t) + \mathcal{F}_2(t),$$

where N and N_1 are positive constants to be determined later.

The differentiation of $\mathcal{L}(t)$ and the use of (7), (41) and (43) lead to

$$\mathcal{L}'(t) \leq -\delta N \int_0^L \theta_x^2 dx - \frac{\mu N_1}{2} \int_0^L \varphi_t^2 dx + \varepsilon N_1 \int_0^L (\varphi_x + \psi)^2 dx$$
$$+ N_1 \left(\frac{\delta^2}{2\mu} + \frac{c\mu c_P}{\rho} + \frac{c^2 \kappa^2 c_P}{4\varepsilon \rho^2} \right) \int_0^L \theta_x^2 dx$$
$$- \frac{b}{2} \int_0^L \psi_x^2 dx - \frac{\kappa}{2} \int_0^L (\varphi_x + \psi)^2 dx + \rho \int_0^L \varphi_t^2 dx + M \int_0^L \theta_x^2 dx;$$

that is,

$$\mathcal{L}'(t) \leq - \left(\frac{\mu N_1}{2} - \rho \right) \int_0^L \varphi_t^2 dx - \left(\frac{\kappa}{2} - \varepsilon N_1 \right) \int_0^L (\varphi_x + \psi)^2 dx - \frac{b}{2} \int_0^L \psi_x^2 dx$$
$$- \left(\delta N - N_1 \left(\frac{\delta^2}{2\mu} + \frac{c\mu c_P}{\rho} + \frac{c^2 \kappa^2 c_P}{4\varepsilon \rho^2} \right) - M \right) \int_0^L \theta_x^2 dx.$$

At this stage, we choose $N_1 > \dfrac{\rho}{2\mu}$, then select $\varepsilon < \dfrac{\kappa}{2N_1}$, and finally, pick N large such that

$$\delta N - N_1 \left(\frac{\delta^2}{2\mu} + \frac{c\mu c_P}{\rho} + \frac{c^2 \kappa^2 c_P}{4\varepsilon \rho^2} \right) - M > 0.$$

Thus, we deduce that there exists $\chi > 0$, for which

$$\mathcal{L}'(t) \leq -\chi \int_0^L \left(\varphi_t^2 + \psi_x^2 + (\varphi_x + \psi)^2 + \theta_x^2 \right) dx. \tag{46}$$

Recalling Poincaré's inequality, we easily show that

$$\mathcal{L}'(t) \leq -\xi E(t),$$

for some positive constant ξ.

On the other hand, one can choose N large enough such that $\mathcal{L}(t) \sim E(t)$ and (46) remains true. Indeed, we have

$$|\mathcal{L}(t) - NE(t)| = N_1 c \int_0^L \varphi_t \int_0^x \theta(y) dy dx + \rho \int_0^L \varphi_t \varphi dx.$$

Cauchy–Schwarz, Young's and Poincaré's inequalities give

$$|\mathcal{L}(t) - NE(t)| \leq \lambda E(t), \, t \geq 0,$$

for $\lambda > 0$. Consequently,

$$(N - \lambda) E(t) \leq \mathcal{L}(t) \leq (N + \lambda) E(t), \forall t \geq 0.$$

Thus, for $N > \lambda$, we have $\mathcal{L}(t) \sim E(t)$ and therefore, for

$$N \geq \max \left\{ \lambda, N_1 \left(\frac{\delta^2}{2\mu} + \frac{c\mu c_P}{\rho} + \frac{c^2 \kappa^2 c_P}{4\varepsilon \rho^2} \right) + M \right\},$$

there exists $\gamma > 0$, such that

$$\mathcal{L}'(t) \leq -\gamma \mathcal{L}(t), \forall t \geq 0.$$

By integrating the last inequality with respect to t over $(0,t)$, we obtain

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\gamma t}, \forall t \geq 0.$$

The use of the equivalence between $\mathcal{L}(t)$ and $E(t)$ leads to (39). The proof of Theorem 3.1 is then completed.

4 Numerical approximation

In this section, we propose a finite element approximation for the system (3) with the initial conditions (4) and subject to the boundary conditions (5). Furthermore, we demonstrate that the discrete energy decays, leading to the derivation of a discrete stability property.

Multiplying (3) by test functions ξ, η, $\zeta \in H_0^1(0,L)$, we get the following weak form:

$$\begin{cases} \rho(v_t,\xi) + \kappa(\varphi_x + \psi, \xi_x) - \mu(\theta, \xi_x) = 0, \\ b(\psi_x, \eta_x) + \kappa(\varphi_x + \psi, \eta) = 0, \\ c(\theta_t, \zeta) + \delta(\theta_x, \zeta_x) + \mu(v_x, \zeta) = 0, \end{cases} \tag{47}$$

where $v = \varphi_t$, and $(\cdot,\cdot)$ is the L^2 inner product.

Let us partition the interval $(0,L)$ into subintervals $I_i = (x_{i-1}, x_i)$ of length $h = 1/m$ with $0 = x_0 < x_1 < \cdots < x_m = L$ and define

$$P_h^1 = \left\{ u \in H_0^1(0,L), u \mid_{I_i} \text{ is a linear polynomial} \right\}.$$

For a given final time T and a positive integer N, let $\Delta t = T/N$ be the time step and $t_n = n\Delta t$, $n = 0, \ldots, N$. The finite element method for (47) is to find v_h^n, ψ_h^n, $\theta_h^n \in P_h^1$, such that, for all ξ_h, η_h, $\zeta_h \in P_h^1$,

$$\begin{cases} \dfrac{\rho}{\Delta t}(v_h^n - v_h^{n-1}, \xi_h) + \kappa(\varphi_{hx}^n + \psi_h^n, \xi_{hx}) - \mu(\theta_h^n, \xi_{hx}) = 0, \\ b(\psi_{hx}^n, \eta_{hx}) + \kappa(\varphi_{hx}^n + \psi_h^n, \eta_h) = 0, \\ \dfrac{c}{\Delta t}(\theta_h^n - \theta_h^{n-1}, \zeta_h) + \delta(\theta_{hx}^n, \zeta_{hx}) + \mu(v_{hx}^n, \zeta_h) = 0, \end{cases} \tag{48}$$

with $\varphi_h^n = \varphi_h^{n-1} + \Delta t v_h^n$ and the notations φ_h^0, v_h^0, ψ_h^0, θ_h^0 are adequate approximations to φ_0, φ_1, ψ_0 and θ_0, respectively. Then, the discrete energy is given by

$$E_h^n = \frac{1}{2}\left(\rho \|v_h^n\|_2^2 + b\|\psi_{hx}^n\|_2^2 + c\|\theta_h^n\|_2^2 + \kappa \|\varphi_{hx}^n + \psi_h^n\|_2^2 \right). \tag{49}$$

The next result is a discrete version of the energy decay property satisfied by the solution of system (3).

Theorem 4.1. *For $n = 1, 2, \ldots, N$, the discrete energy satisfies the following stability property*

$$E_h^n - E_h^{n-1} \leq 0. \tag{50}$$

Proof. Choosing $\xi_h = v_h^n$, $\eta_h = \psi_h^n$, and $\zeta_h = \theta_h^n$ in (48) and thanks to the following identity

$$(a-b,a) = \frac{1}{2}\left(\|a-b\|_2^2 + \|a\|_2^2 - \|b\|_2^2\right),$$

we obtain

$$\begin{cases} \frac{\rho}{2\Delta t}(\|v_h^n - v_h^{n-1}\|_2^2 + \|v_h^n\|_2^2 - \|v_h^{n-1}\|_2^2) + \kappa(\varphi_{hx}^n + \psi_h^n, v_{hx}^n) - \mu(\theta_h^n, v_{hx}^n) = 0, \\ b\|\psi_{hx}^n\|_2^2 + \kappa(\varphi_{hx}^n + \psi_h^n, \psi_h^n) = 0, \\ \frac{c}{2\Delta t}(\|\theta_h^n - \theta_h^{n-1}\|_2^2 + \|\theta_h^n\|_2^2 - \|\theta_h^{n-1}\|_2^2) + \delta\|\theta_{hx}^n\|_2^2 + \mu(v_{hx}^n, \theta_h^n) = 0. \end{cases} \tag{51}$$

Adding the latter equations, we find that

$$\begin{aligned} &\frac{\rho}{2\Delta t}(\|v_h^n - v_h^{n-1}\|_2^2 + \|v_h^n\|_2^2 - \|v_h^{n-1}\|_2^2) + \kappa(\varphi_{hx}^n + \psi_h^n, v_{hx}^n + \psi_h^n) + b\|\psi_{hx}^n\|_2^2 \\ &+ \frac{c}{2\Delta t}(\|\theta_h^n - \theta_h^{n-1}\|_2^2 + \|\theta_h^n\|_2^2 - \|\theta_h^{n-1}\|_2^2) + \delta\|\theta_{hx}^n\|_2^2 = 0. \end{aligned}$$

We note that, from Young's inequality and the second equation of (3), we have

$$\begin{aligned} \kappa(\varphi_{hx}^n + \psi_h^n, v_{hx}^n + \psi_h^n) =& \kappa(\varphi_{hx}^n + \psi_h^n, \varphi_{hx}^n + \psi_h^n - \varphi_{hx}^n + v_{hx}^n) \\ =& \kappa\|\varphi_{hx}^n + \psi_h^n\|_2^2 + \kappa(\varphi_{hx}^n + \psi_h^n, -\varphi_{hx}^{n-1} + (1-\Delta t)v_{hx}^n) \\ =& \kappa\|\varphi_{hx}^n + \psi_h^n\|_2^2 - \kappa(\varphi_{hx}^n + \psi_h^n, \varphi_{hx}^{n-1} + \psi_h^{n-1}) \\ &+ \kappa(\varphi_{hx}^n + \psi_h^n, \psi_h^{n-1} + (1-\Delta t)v_{hx}^n), \\ \geq& \frac{\kappa}{2}\|\varphi_{hx}^n + \psi_h^n\|_2^2 - \frac{\kappa}{2}\|\varphi_{hx}^{n-1} + \psi_h^{n-1}\|_2^2 + b(\psi_{hxx}^n, \psi_h^{n-1}) \\ &+ (1-\Delta t)\kappa(\varphi_{hx}^n + \psi_h^n, v_{hx}^n). \end{aligned}$$

Therefore

$$\begin{aligned} \kappa(\varphi_{hx}^n + \psi_h^n, v_{hx}^n + \psi_h^n) \geq& \frac{\kappa}{2}\|\varphi_{hx}^n + \psi_h^n\|_2^2 - \frac{\kappa}{2}\|\varphi_{hx}^{n-1} + \psi_h^{n-1}\|_2^2 - \frac{b}{2}\|\psi_{hx}^n\|_2^2 \\ &- \frac{b}{2}\|\psi_{hx}^{n-1}\|_2^2 + (1-\Delta t)\kappa(\varphi_{hx}^n + \psi_h^n, v_{hx}^n). \end{aligned}$$

Next, from (51), we have

$$\begin{aligned} (1-\Delta t)\kappa(\varphi_{hx}^n + \psi_h^n, v_{hx}^n) =& (1 - \frac{1}{\Delta t})\rho(v_h^n - v_h^{n-1}, v_h^n) + \mu(1-\Delta t)(\theta_h^n, v_{hx}^n) \\ =& (1 - \frac{1}{\Delta t})\frac{\rho}{2}\left(\|v_h^n - v_h^{n-1}\|_2^2 + \|v_h^n\|_2^2 - \|v_h^{n-1}\|_2^2\right) \\ &+ (1 - \frac{1}{\Delta t})c(\theta_h^n - \theta_h^{n-1}, \theta_h^n) - (1-\Delta t)\delta\|\theta_{hx}^n\|_2^2 \\ =& (1 - \frac{1}{\Delta t})\frac{\rho}{2}\left(\|v_h^n - v_h^{n-1}\|_2^2 + \|v_h^n\|_2^2 - \|v_h^{n-1}\|_2^2\right) \\ &+ (1 - \frac{1}{\Delta t})\frac{c}{2}\left(\|\theta_h^n - \theta_h^{n-1}\|_2^2 + \|\theta_h^n\|_2^2 - \|\theta_h^{n-1}\|_2^2\right) \\ &- (1-\Delta t)\delta\|\theta_{hx}^n\|_2^2. \end{aligned}$$

These results together yield

$$\begin{aligned}&\frac{1}{2}\left(\rho\|v_h^n\|_2^2+b\|\psi_{hx}^n\|_2^2+c\|\theta_h^n\|_2^2+\kappa\|\varphi_{hx}^n+\psi_h^n\|_2^2\right)\\ -&\frac{1}{2}\left(\rho\|v_h^{n-1}\|_2^2+b\|\psi_{hx}^{n-1}\|_2^2+c\|\theta_h^{n-1}\|_2^2+\kappa\|\varphi_{hx}^{n-1}+\psi_h^{n-1}\|_2^2\right)\leq 0\end{aligned}$$

and the theorem is proved using the definition of the discrete energy (49). □

5 Numerical experiments

In order to illustrate the theoretical results in Theorem 3.6., some numerical experiments have been performed using the numerical method analysed in the previous section. We divide the spatial interval $[0, L] = [0, 1]$ into $m = 50$ subintervals, where the spatial step size $h = 0.02$. The temporal interval $[0, T] = [0, 50]$ with a time-step size $\Delta t = 10^{-2}$.

We run our code for N time steps ($N = T/\Delta t$), using the following initial conditions:

$$\varphi_h^0(x) = 2\sin\left(\frac{\pi}{2}x\right), \quad v_h^0(x) = \sin\left(\frac{\pi}{2}x\right),$$

$$\psi_h^0(x) = 2\sin\left(\frac{\pi}{2}x\right), \quad \theta_h^0(x) = \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right).$$

The numerical tests are done for different entries as follows:

- Test 1:
$$\rho = \kappa = \mu = \delta = c = b = 1.$$
- Test 2:
$$\rho = \kappa = \delta = c = b = 1 \text{ and } \mu = 0.$$
- Test 3:
$$\rho = 1/3, \kappa = 1/2, \mu = 10^{-2} \text{ and } \delta = c = b = 1.$$

For each numerical test, we plot the cross section cuts for the approximate solution (φ, ψ, θ) at different points ($x = 0.3$, $x = 0.5$ and $x = 0.9$). The numerical results are shown in Figure 1, Figure 2 and Figure 3, successively. Next, we present the energy curves for the three cases in Figure 4 to show the difference between the energy decays according to the parameters chosen in each test.

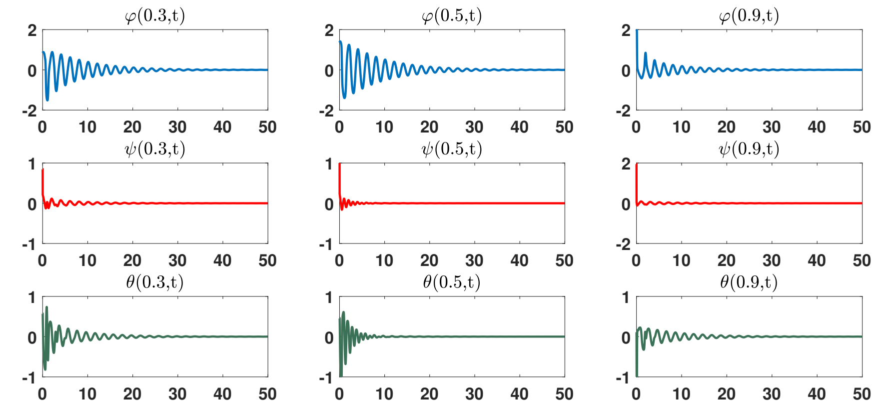

Figure 1: Test 1: Damping cross section waves

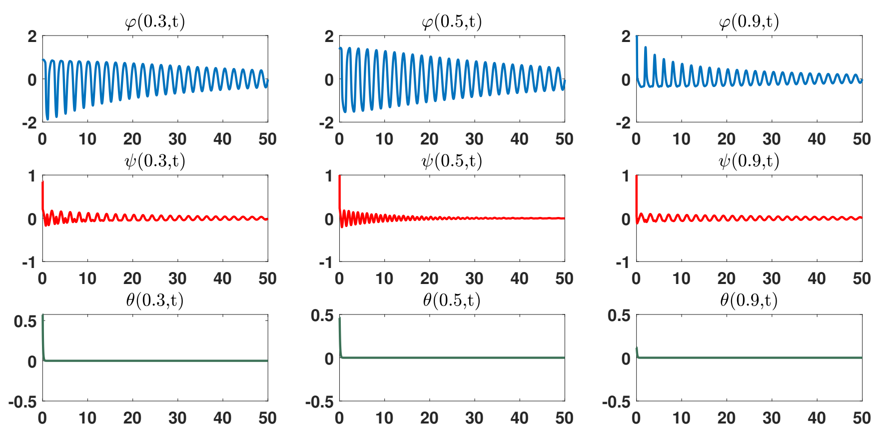

Figure 2: Test 2: Damping cross section waves

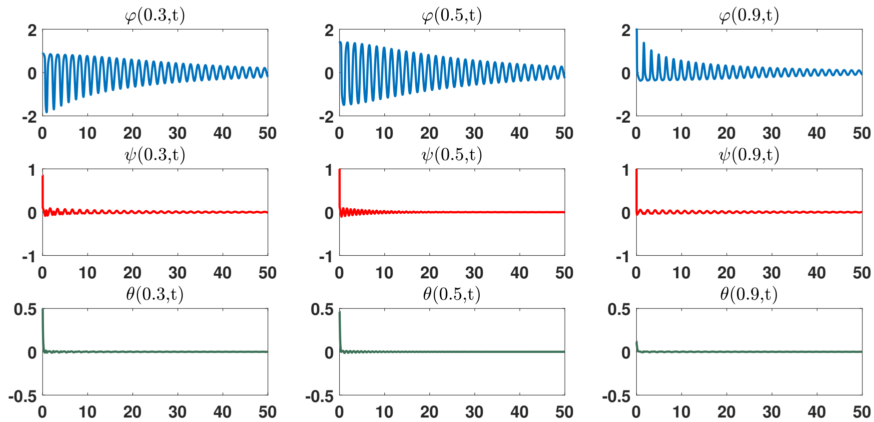

Figure 3: Test 3: Damping cross section waves

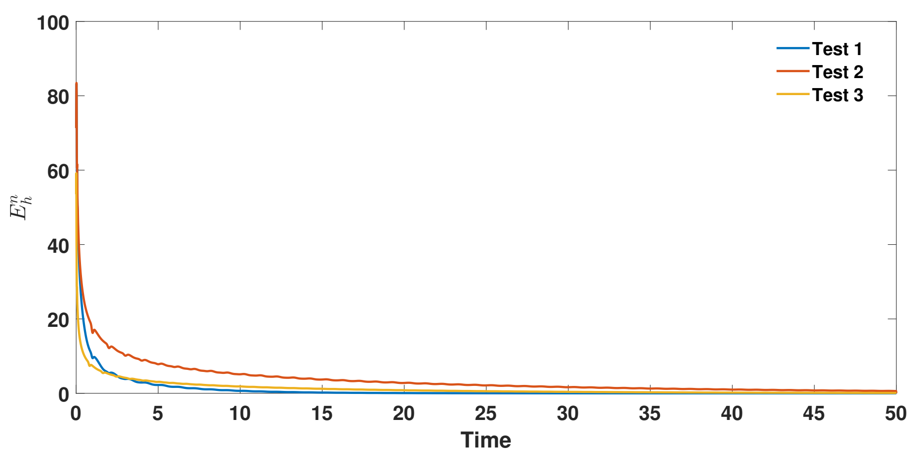

Figure 4: Energy decay

As a conclusion, for all tests, we observed that the numerical solution converges to zero and an exponential decay with different rates seems to be reached which is compatible with the theoretical results, even for the undamped case $\mu = 0$ (Test 2), which has been included for comparison.

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