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U-numbers in fields of positive characteristic

by
Gülcan Kekeç

Dedicated to the memory of Professors Orhan Şerafettin İçen and Kamil Alnıaçık

Abstract

We extend a previous result of Kekeç and prove that under certain conditions rational combinations with algebraic formal power series coefficients of a U_1-number are U_m-numbers in the field of formal power series over a finite field.

Key Words: Mahler's classification of transcendental formal power series over a finite field, U-number, transcendence measure.
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1 Introduction

Let p be a prime number and K be a finite field of characteristic p and with q elements. Denote the ring of polynomials in x with coefficients in K by $K[x]$, the quotient field of $K[x]$ by $K(x)$ and the degree of a non-zero polynomial $a(x)$ in $K[x]$ by $\deg(a)$. Then a non-Archimedean absolute value $|\cdot|$ is set on $K(x)$ by

$$|0| = 0 \quad \text{and} \quad \left|\frac{a(x)}{b(x)}\right| = q^{\deg(a)-\deg(b)},$$

where $a(x)$ and $b(x)$ are non-zero polynomials in $K[x]$. The completion of $K(x)$ with respect to $|\cdot|$ is the field $\mathbb{K}$ of formal power series over K. The absolute value $|\cdot|$ is uniquely extended from $K(x)$ to $\mathbb{K}$ and is denoted by the same notation $|\cdot|$. We uniquely represent each non-zero element ξ of $\mathbb{K}$ as

$$\xi = \sum_{h=l}^{\infty} a_h x^{-h},$$

where $a_h \in K$ for $h = l, l+1, \dots$ with $a_l \neq 0$ and l is the rational integer satisfying $|\xi| = q^{-l}$.

$\xi \in \mathbb{K}$ is called an algebraic formal power series if it is algebraic over $K(x)$ and a transcendental formal power series if it is transcendental over $K(x)$. Let $P(y) = c_0 + c_1 y + \cdots + c_n y^n$ be a non-zero polynomial in y with coefficients in $K[x]$. The degree of $P(y)$ with respect to y is denoted by $\deg(P)$ and the height $H(P)$ of $P(y)$ is defined as $H(P) = \max\{|c_0|, |c_1|, \dots, |c_n|\}$. Let $\alpha \in \mathbb{K}$ be an algebraic formal power series and $P(y)$ be its minimal polynomial over $K[x]$. Then the degree $\deg(\alpha)$ of α is defined as $\deg(P)$ and the height $H(\alpha)$ of α is defined as $H(P)$. (See Sprindžuk [22] for information on $\mathbb{K}$.)

In 1932, Mahler [17] gave a classification of real numbers. He separated transcendental real numbers into three disjoint classes called $S-$, $T-$ and $U-$numbers. The class of

$U-$numbers is subdivided into U_m-subclasses ($m = 1, 2, 3, \ldots$). LeVeque [16] established the first explicit illustrations of U_m-numbers for each positive integer m.

The completion of the field $\mathbb{Q}$ of rational numbers with respect to the $p-$adic absolute value, where p is a prime number, is the field $\mathbb{Q}_p$ of $p-$adic numbers. In 1935, Mahler [18] introduced a classification of $p-$adic numbers similar to his classification of real numbers. He split up transcendental $p-$adic numbers into three disjoint classes called $p-$adic $S-$, $T-$ and $U-$numbers. The class of $p-$adic $U-$numbers is subdivided into U_m-subclasses ($m = 1, 2, 3, \ldots$). Alnıaçık [1, Chapter III, Theorem I] established the first explicit illustrations of $p-$adic U_m-numbers for each positive integer m.

In 1978, Bundschuh [3] proposed a classification in $\mathbb{K}$, similar to Mahler's classification in $\mathbb{R}$ and in $\mathbb{Q}_p$, which is called Mahler's classification of formal power series over a finite field. Bundschuh [3] separated transcendental formal power series into three disjoint classes as follows.

Let $\xi \in \mathbb{K}$ be a transcendental formal power series. For positive rational integers n and H, define

$$w_n(H, \xi) = \min \{|P(\xi)| : P(y) \in K[x][y] \setminus \{0\},\ \deg(P) \leq n \text{ and } H(P) \leq H\},$$

$$w_n(\xi) = \limsup_{H \to \infty} \frac{-\log w_n(H, \xi)}{\log H} \quad \text{and} \quad w(\xi) = \limsup_{n \to \infty} \frac{w_n(\xi)}{n}.$$

Bundschuh [3] proved that $w(\xi) \geq 1$. Then ξ is called

- an $S-$number if $1 \leq w(\xi) < \infty$,
- a $T-$number if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ ($n = 1, 2, 3, \ldots$),
- a $U-$number if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ from some n onward.

Furthermore, a $U-$number ξ is said to be a U_1-number if $w_1(\xi) = \infty$ and a U_m-number if $w_m(\xi) = \infty$ and $w_n(\xi) < \infty$ for $n = 1, \ldots, m-1$, where $m > 1$. Oryan [20] gave the first explicit constructions of U_m-numbers for each positive integer m. Recently, [12], [13], [14], [4] and [5] established further explicit illustrations of U_m-numbers in $\mathbb{K}$. (See Bugeaud [2] for a detailed information on Mahler's classification in $\mathbb{R}$, in $\mathbb{Q}_p$, in $\mathbb{K}$ and Lasjaunias [15] for a survey of Diophantine approximation in fields of power series.)

2 Our main result

In 1979, Alnıaçık [1, Chapter III, Theorem I, pages 73-81] proved the existence of $p-$adic U_m-numbers for each positive integer m by showing that under certain conditions rational combinations with $p-$adic algebraic coefficients of a $p-$adic U_1-number are $p-$adic U_m-numbers. Recently, Kekeç [14] established the following analogue of Alnıaçık [1, Chapter III, Theorem I] in the field $\mathbb{K}$ when the combination is integral.

Theorem 1 (Kekeç [14], Theorem 1.1)**.** *Let $\alpha_0, \ldots, \alpha_k$ ($k \geq 1$) be algebraic formal power series with $\alpha_k \neq 0$ and ξ be a U_1-number enjoying a representation of the form*

$$\xi = \sum_{n=0}^{\infty} a_n x^{-u_n},$$

where $a_n \in K \setminus \{0\}$ $(n = 0, 1, 2, \ldots)$ *and* $\{u_n\}_{n=0}^{\infty}$ *is a strictly increasing sequence of non-negative rational integers with*

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \infty.$$

Then $\alpha_0 + \alpha_1\xi + \cdots + \alpha_k\xi^k$ *is a* U_m-*number, where* m *is the degree of* $K(x)(\alpha_0, \ldots, \alpha_k)$ *over* $K(x)$.

In the present paper, in Theorem 2, we extend Theorem 1 from integral combination to rational combination. Therefore, in the field $\mathbb{K}$ of formal power series, we establish the exact analogue of Alnıaçık [1, Chapter III, Theorem I] by using the recent result Can and Kekeç [4, Theorem 1.2].

Theorem 2. *Let* $\alpha_0, \ldots, \alpha_k$ $(k \geq 1)$ *and* $\beta_0, \ldots, \beta_l$ $(l \geq 1)$ *be algebraic formal power series with* $\alpha_k \neq 0$ *and* $\beta_l = 1$. *Let* m *be the degree of* $K(x)(\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_l)$ *over* $K(x)$. *Moreover, suppose that the polynomials* $C(y) = \alpha_0 + \alpha_1 y + \cdots + \alpha_k y^k$ *and* $D(y) = \beta_0 + \beta_1 y + \cdots + \beta_l y^l$ *are relatively prime over*

$$K(x)(\alpha_0^{\{1\}}, \ldots, \alpha_0^{\{m\}}, \ldots, \alpha_k^{\{1\}}, \ldots, \alpha_k^{\{m\}}, \beta_0^{\{1\}}, \ldots, \beta_0^{\{m\}}, \ldots, \beta_l^{\{1\}}, \ldots, \beta_l^{\{m\}}),$$

where $\alpha_i^{\{1\}}, \ldots, \alpha_i^{\{m\}}$ *and* $\beta_j^{\{1\}}, \ldots, \beta_j^{\{m\}}$ *denote the field conjugates of* α_i $(i = 0, 1, \ldots, k)$ *and* β_j $(j = 0, 1, \ldots, l)$ *for* $K(x)(\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_l)$, *respectively. Assume that* ξ *is a* U_1-*number enjoying a representation of the form*

$$\xi = \sum_{h=0}^{\infty} a_h x^{-h},$$

where $a_h \in K$ $(h = 0, 1, 2, \ldots)$ *satisfying*

$$\begin{cases} a_h = 0, & r_n < h < s_n \quad (n = 1, 2, 3, \ldots), \\ a_h \neq 0, & h = r_n \qquad\quad\; (n = 1, 2, 3, \ldots), \\ a_h \neq 0, & h = s_n \qquad\quad\; (n = 0, 1, 2, \ldots), \end{cases}$$

where $\{s_n\}_{n=0}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *are two infinite sequences of non-negative rational integers with*

$$0 = s_0 \leq r_1 < s_1 \leq r_2 < s_2 \leq r_3 < s_3 \leq r_4 < s_4 \leq \ldots,$$

$$\lim_{n\to\infty} \frac{s_n}{r_n} = \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{r_{n+1}}{s_n} < \infty.$$

Then

$$\gamma := \frac{C(\xi)}{D(\xi)} = \frac{\alpha_0 + \alpha_1\xi + \cdots + \alpha_k\xi^k}{\beta_0 + \beta_1\xi + \cdots + \beta_l\xi^l}$$

is a U_m-*number.*

In the next section, we prepare and cite some auxiliary results to prove Theorem 2. We prove Theorem 2 in Section 4 and construct U_m-numbers by applying Theorem 2 in Section 5.

3 Auxiliary results

The following lemma, Lemma 1, is the exact analogue of Alnıaçık [1, Chapter I, Lemma 5] in the power series setting and is an extension of Kekeç [14, Lemma 2.1] from integral combination to rational combination. The proof of Lemma 1 is the same as that of Alnıaçık [1, Chapter I, Lemma 5]. Therefore, we omit the proof of Lemma 1.

Lemma 1. *Let $\alpha_0, \dots, \alpha_k$ $(k \geq 1)$ and $\beta_0, \dots, \beta_l$ $(l \geq 1)$ be algebraic formal power series with $\alpha_k \neq 0$ and $\beta_l = 1$. Let m be the degree of $K(x)(\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_l)$ over $K(x)$. Moreover, suppose that the polynomials $C(y) = \alpha_0 + \alpha_1 y + \cdots + \alpha_k y^k$ and $D(y) = \beta_0 + \beta_1 y + \cdots + \beta_l y^l$ are relatively prime over*

$$K(x)(\alpha_0^{\{1\}}, \dots, \alpha_0^{\{m\}}, \dots, \alpha_k^{\{1\}}, \dots, \alpha_k^{\{m\}}, \beta_0^{\{1\}}, \dots, \beta_0^{\{m\}}, \dots, \beta_l^{\{1\}}, \dots, \beta_l^{\{m\}}).$$

Then for θ in $K(x)$ the algebraic formal power series

$$\frac{C(\theta)}{D(\theta)} = \frac{\alpha_0 + \alpha_1\theta + \cdots + \alpha_k\theta^k}{\beta_0 + \beta_1\theta + \cdots + \beta_l\theta^l}$$

is a primitive element of $K(x)(\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_l)$ over $K(x)$ except for only finitely many θ in $K(x)$.

The following recent result Can and Kekeç [4, Theorem 1.2], which is a power series analogue of the results İçen [11, page 25] and [10, Lemma 1, page 71], plays an important role in the proof of Theorem 2.

Theorem 3 (Can and Kekeç [4], Theorem 1.2)**.** *Let L be a finite extension of degree m over $K(x)$ and $\alpha_1, \alpha_2, \dots, \alpha_k$ be in L. Let η be any algebraic formal power series. Assume that $F(\eta, \alpha_1, \dots, \alpha_k) = 0$, where $F(y, y_1, \dots, y_k)$ is a polynomial in $y, y_1, \dots, y_k$ over $K[x]$ with degree at least 1 in y. Then*

$$H(\eta) \leq H^m H(\alpha_1)^{l_1 m} \cdots H(\alpha_k)^{l_k m},$$

where l_j is the degree of $F(y, y_1, \dots, y_k)$ in y_j $(j = 1, \dots, k)$ and H is the maximum of the absolute values of the coefficients of $F(y, y_1, \dots, y_k)$.

The following lemma is a well-known result the proof of which can be done easily by following the lines of Waldschmidt [23, 3.5 Liouville's Inequalities, page 82].

Lemma 2. *Let α be an algebraic formal power series. Then*

$$|\alpha| \leq H(\alpha).$$

Lemma 3 (Oryan [20], Hilfssatz 2)**.** *Let $P(y)$ and $Q(y)$ be polynomials over $K[x]$ with degrees $n \geq 1$ and $m \geq 2$, respectively. If the polynomials $P(y)$ and $Q(y)$ are relatively prime over $K[x]$ and α is a root of $Q(y)$, then*

$$|P(\alpha)| \geq H(P)^{-m+1} H(Q)^{-n}.$$

4 Proof of Theorem 2

We prove Theorem 2 by making use of the methods of the proofs of Alnıaçık [1, Chapter III, Theorem I], Oryan [20, Satz 4, Satz 5] and Kekeç [14, Theorem 1.1]. We have

$$\xi = \xi_n + \rho_n \qquad (n = 1, 2, 3, \ldots),$$

where

$$\xi_n = \sum_{h=s_0}^{r_n} a_h x^{-h} \quad \text{and} \quad \rho_n = \sum_{h=s_n}^{\infty} a_h x^{-h} \qquad (n = 1, 2, 3, \ldots).$$

Therefore,

$$|\xi| = |\xi_n| = q^{-s_0} = 1 \qquad (n = 1, 2, 3, \ldots) \tag{4.1}$$

and

$$|\rho_n| = q^{-s_n} < 1 \qquad (n = 1, 2, 3, \ldots). \tag{4.2}$$

Then

$$C(\xi) = C(\xi_n + \rho_n) = C(\xi_n) + \rho_n \delta_n$$

and

$$D(\xi) = D(\xi_n + \rho_n) = D(\xi_n) + \rho_n \widetilde{\delta_n},$$

where

$$\delta_n = \alpha_1 + \alpha_2(2\xi_n + \rho_n) + \cdots + \alpha_k \left(\binom{k}{1} \xi_n^{k-1} + \binom{k}{2} \xi_n^{k-2} \rho_n + \cdots + \rho_n^{k-1} \right)$$

and

$$\widetilde{\delta_n} = \beta_1 + \beta_2(2\xi_n + \rho_n) + \cdots + \beta_l \left(\binom{l}{1} \xi_n^{l-1} + \binom{l}{2} \xi_n^{l-2} \rho_n + \cdots + \rho_n^{l-1} \right)$$

for $n = 1, 2, 3, \ldots$. As the equation $D(y) = 0$ may only have finitely many solutions in $K(x)$, the relation $D(\xi_n) \neq 0$ holds for sufficiently large n. Thus we get for sufficiently large n

$$\gamma = \gamma_n + \rho_n \sigma_n,$$

where

$$\gamma_n = \frac{C(\xi_n)}{D(\xi_n)} \quad \text{and} \quad \sigma_n = \frac{D(\xi_n)\delta_n - C(\xi_n)\widetilde{\delta_n}}{D(\xi_n) D(\xi)}.$$

Let

$$|\alpha_i| =: q^{-e_i} \quad (i = 0, 1, \ldots, k), \qquad |\beta_j| =: q^{-f_j} \quad (j = 0, 1, \ldots, l)$$

and

$$e := \min\{0, e_0, e_1, \ldots, e_k\}, \qquad f := \min\{0, f_0, f_1, \ldots, f_l\}, \qquad t := \max\{0, f_0, f_1, \ldots, f_l\}.$$

Hence, using (4.1) and (4.2), we obtain for sufficiently large n

$$|\delta_n| \leq q^{-e}, \qquad |\widetilde{\delta_n}| \leq q^{-f} \qquad \text{and} \qquad |\sigma_n| \leq q^{2t-e-f}. \tag{4.3}$$

Observe that $\gamma_n \in K(x)(\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_l)$. By Lemma 1, $\deg(\gamma_n) = m$ holds for sufficiently large n. We have

$$F(\gamma_n, \alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_l) = 0,$$

where

$$F(y, y_0, \ldots, y_{k+l+1}) = \left(yy_{k+1} + \xi_n yy_{k+2} + \cdots + \xi_n^l yy_{k+l+1} - y_0 - \xi_n y_1 - \cdots - \xi_n^k y_k\right) x^{r_n \max\{k,l\}}$$

is a polynomial in $K[x][y, y_0, \ldots, y_{k+l+1}]$. So it follows from Theorem 3 that

$$H(\gamma_n) \leq q^{r_n \max\{k,l\}m} H(\alpha_0)^m \cdots H(\alpha_k)^m H(\beta_0)^m \cdots H(\beta_l)^m \leq q^{r_n(\max\{k,l\}m+1)} \tag{4.4}$$

is satisfied for sufficiently large n.

Let $P_n(y) = c_{n0} + c_{n1}y + \cdots + c_{nm}y^m$ be the minimal polynomial of γ_n over $K[x]$. For $P_n(\gamma_n) = 0$, we get

$$P_n(\gamma) = P_n(\gamma_n + \rho_n\sigma_n) = \rho_n\theta_n,$$

where

$$\theta_n = c_{n1}\sigma_n + \cdots + c_{nm}\left(\binom{m}{1}\gamma_n^{m-1}\sigma_n + \binom{m}{2}\gamma_n^{m-2}\rho_n\sigma_n^2 + \cdots + \rho_n^{m-1}\sigma_n^m\right).$$

We infer from Lemma 2, (4.2), (4.3) and (4.4) that

$$|\theta_n| \leq H(\gamma_n)^m |\sigma_n|^m \leq d_1^{r_n},$$

where $d_1 = q^{m^2 \max\{k,l\}+m+2tm-em-fm}$, so from (4.2) that

$$|P_n(\gamma)| = |\rho_n||\theta_n| \leq q^{-s_n} d_1^{r_n}$$

and therefore from (4.4) and the fact $H(P_n) = H(\gamma_n)$ that

$$0 < |P_n(\gamma)| \leq H(P_n)^{-\lambda_n}$$

are verified for sufficiently large n, where

$$\lambda_n = \frac{s_n}{r_n}\frac{\log q}{\log d_1} - 1 \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \infty.$$

Since $\deg(P_n) = m$ holds, this implies that γ is a $U-$number with

$$w_m(\gamma) = \infty. \tag{4.5}$$

If $m = 1$, then $w_1(\gamma) = \infty$ and so γ is a U_1-number. Thus assume that $m > 1$. Let $B(y) = b_0 + b_1y + \cdots + b_gy^g$ be any polynomial over $K[x]$ with $\deg(B) = g$, $1 \leq g \leq m-1$ and with sufficiently large height $H(B)$. For any sufficiently large positive rational integer ν,

$$B(\gamma) = B(\gamma_\nu + \rho_\nu\sigma_\nu) = B(\gamma_\nu) + \rho_\nu\varphi_\nu, \tag{4.6}$$

where

$$\varphi_\nu = b_1\sigma_\nu + \cdots + b_g\left(\binom{g}{1}\gamma_\nu^{g-1}\sigma_\nu + \binom{g}{2}\gamma_\nu^{g-2}\rho_\nu\sigma_\nu^2 + \cdots + \rho_\nu^{g-1}\sigma_\nu^g\right).$$

Therefore, we get for sufficiently large ν

$$|\varphi_\nu| \le d_1^{r_\nu} \quad \text{and} \quad |\rho_\nu||\varphi_\nu| \le d_1^{-r_\nu \lambda_\nu}.$$

There is a real constant d_2 with $0 < d_2 < 1$ such that

$$\lambda_\nu \ge d_2 \frac{s_\nu}{r_\nu}$$

and so

$$|\rho_\nu||\varphi_\nu| \le d_1^{-d_2 s_\nu} \tag{4.7}$$

hold for sufficiently large ν. Since $\deg(\gamma_\nu) = m$, we have $B(\gamma_\nu) \ne 0$. Hence, by Lemma 3 and (4.4), we obtain for sufficiently large ν

$$|B(\gamma_\nu)| \ge H(B)^{-(m-1)} H(\gamma_\nu)^{-(m-1)} \ge H(B)^{-(m-1)} d_1^{-r_\nu(m-1)}. \tag{4.8}$$

As $\limsup_{n\to\infty} \frac{r_{n+1}}{s_n} < \infty$, there exists a real number $\mu > 1$ such that

$$\frac{r_{n+1}}{s_n} < \mu \tag{4.9}$$

for sufficiently large n. Let ϕ and ψ be two real numbers satisfying

$$\phi > \frac{2(m-1)\mu}{d_2} \tag{4.10}$$

and

$$\psi > \frac{(m-1)(\phi+1)}{d_2}. \tag{4.11}$$

Further, the inequality

$$\psi < \frac{s_\nu}{r_\nu} \tag{4.12}$$

holds for sufficiently large ν.

Let n be the unique positive rational integer verifying

$$d_1^{r_n} \le H(B) < d_1^{r_{n+1}}.$$

If $d_1^{r_n} \le H(B) < d_1^{r_{n+1}/\phi}$ holds, then we deduce from (4.6), (4.7), (4.8) and (4.9) for $\nu = n$ that

$$B(\gamma) = B(\gamma_n) + \rho_n \varphi_n,$$
$$|B(\gamma_n)| \ge H(B)^{-2(m-1)},$$
$$|\rho_n||\varphi_n| < H(B)^{-(d_2\phi)/\mu}.$$

So, by (4.10), it follows that $|\rho_n||\varphi_n| < |B(\gamma_n)|$. Thus we obtain

$$|B(\gamma)| = |B(\gamma_n)| \ge H(B)^{-2(m-1)}. \tag{4.13}$$

If $d_1^{r_{n+1}/\phi} \le H(B) < d_1^{r_{n+1}}$ holds, then we infer from (4.6), (4.7), (4.8) and (4.12) for $\nu = n+1$ that

$$B(\gamma) = B(\gamma_{n+1}) + \rho_{n+1}\varphi_{n+1},$$

$$|B(\gamma_{n+1})| \geq H(B)^{-(m-1)(\phi+1)},$$
$$|\rho_{n+1}||\varphi_{n+1}| < H(B)^{-\psi d_2}.$$

Hence, by (4.11), it follows that $|\rho_{n+1}||\varphi_{n+1}| < |B(\gamma_{n+1})|$. Therefore,

$$|B(\gamma)| = |B(\gamma_{n+1})| \geq H(B)^{-(m-1)(\phi+1)}. \tag{4.14}$$

We see from (4.13) and (4.14) that

$$|B(\gamma)| \geq H(B)^{-(m-1)(\phi+1)}$$

holds for all polynomials $B(y)$ over $K[x]$ with $1 \leq \deg(B) \leq m-1$ and with sufficiently large height $H(B)$. This gives us

$$w_n(\gamma) < \infty \qquad (n = 1, \ldots, m-1). \tag{4.15}$$

Then, using (4.5) and (4.15), we conclude that

$$\gamma = \frac{C(\xi)}{D(\xi)} = \frac{\alpha_0 + \alpha_1\xi + \cdots + \alpha_k\xi^k}{\beta_0 + \beta_1\xi + \cdots + \beta_l\xi^l}$$

is a U_m-number.

5 Applications of Theorem 2

Mahler [19, Theorem 2] observed that

$$\alpha := x^{-1} + x^{-p} + x^{-p^2} + x^{-p^3} + \cdots$$

is an algebraic formal power series of degree p, which is a root of the polynomial

$$P(y) = xy^p - xy + 1 \in K[x][y].$$

We establish the following example for Theorem 2 by using Mahler's algebraic formal power series α.

Example 1. *In Theorem 2, let us take $k = 1$, $\alpha_0 = 1$, $\alpha_1 = \alpha$, $l = 2$, $\beta_0 = -x$, $\beta_1 = 0$, $\beta_2 = 1$ and the U_1-number ξ as*

$$\xi = \sum_{h=0}^{\infty} a_h x^{-h}$$

with

$$\begin{cases} a_h = 0, & r_n < h < s_n \quad (n = 1, 2, 3, \ldots), \\ a_h = 1, & s_n \leq h \leq r_{n+1} \quad (n = 0, 1, 2, \ldots), \end{cases}$$

where

$$s_0 = 0, \quad s_n = (n+3)^{n+2} \quad \text{and} \quad r_n = 2 \cdot (n+2)^{n+1} \quad (n = 1, 2, 3, \ldots).$$

Then all the hypotheses of Theorem 2 are satisfied, and so

$$\gamma = \frac{\alpha\xi + 1}{\xi^2 - x}$$

is a U_p-number.

As the classical theory of regular continued fractions in the field $\mathbb{R}$ of real numbers, there exists an analogous theory of regular continued fractions in the field $\mathbb{K}$ of formal power series. (For example, see [21], [6] and [4].) It is well-known that the regular continued fraction

$$\beta := [x, x, x, \ldots] \in \mathbb{K}$$

is an algebraic formal power series of degree 2, which is a root of the polynomial

$$P(y) = y^2 - xy - 1 \in K[x][y].$$

We construct the following example for Theorem 2 by using the algebraic regular continued fraction β.

Example 2. *In Example 1, let us replace α by β. Then*

$$\gamma = \frac{\beta\xi + 1}{\xi^2 - x}$$

is a U_2-number.

Christol [7] characterized algebraic formal power series over K in terms of automata. He proved that a formal power series

$$\sum_{h=0}^{\infty} a_h x^{-h}$$

over K is algebraic if and only if the sequence $\{a_h\}_{h=0}^{\infty}$ is $q-$automatic. (We refer the reader to Eilenberg [8] for the information on automata and $q-$automatic sequences.) The result of Christol [7] enables us to obtain the somewhat wider applicability of our result Theorem 2. In general, it is difficult to say anything sharp on the degree of the algebraic formal power series arising from a concrete automaton, but nonetheless it gives us more examples of U_m-numbers even if we can not specify m precisely. For instance, let us take $K = \mathbb{F}_2$, the finite field with two elements, and denote by $\mathbb{F}_2((x^{-1}))$ the field of formal power series over $\mathbb{F}_2$. The so-called Thue-Morse-Hedlund sequence

$$\{a_h\}_{h=0}^{\infty} = 01101001100\ldots$$

is $2-$automatic. Then

$$\theta := \sum_{h=0}^{\infty} a_h x^{-h} \in \mathbb{F}_2((x^{-1}))$$

is an algebraic formal power series, which is in particular a root of the polynomial

$$P(y) = (x+1)^3 y^2 + x(x+1)y + 1 \in \mathbb{F}_2[x][y].$$

(See Christol [7] and Firicel [9].) We finally give the following example for Theorem 2 by using the algebraic formal power series θ of degree 2.

Example 3. *In Example 1, let us replace α by θ. Then*

$$\gamma = \frac{\theta\xi + 1}{\xi^2 - x}$$

is a U_2-number in $\mathbb{F}_2((x^{-1}))$.

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Istanbul University, Faculty of Science, Department of Mathematics,
34134 Vezneciler, Fatih, Istanbul, Turkey
E-mail: `gulkekec@istanbul.edu.tr`