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Some Lucas-like congruences for q-trinomial coefficients

by

Yifan Chen(1), Xiaoxia Wang(2)

Abstract

In this paper, we present several new q-congruences on the q-trinomial coefficients introduced by Andrews and Baxter. As a conclusion, we obtain the following congruence:

$$\left(\!\binom{ap+b}{cp+d}\!\right) \equiv \left(\!\binom{a}{c}\!\right)\left(\!\binom{b}{d}\!\right) + \left(\!\binom{a}{c+1}\!\right)\left(\!\binom{b}{d-p}\!\right) \pmod p,$$

where a, b, c, d are integers subject to $a \geq 0, 0 \leq b, d \leq p-1$, and p is an odd prime.

Besides, we find that the method can also be used to reprove Pan's Lucas-type congruence for the q-Delannoy numbers.

Key Words: q-trinomial coefficients, q-congruences, cyclotomic polynomials.
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1 Introduction

In 1992, Sagan [12] gave the following q-congruence: for $a, b, c, d \in \mathbb{N}$ with $0 \leq b, d \leq n-1$,

$$\begin{bmatrix} an+b \\ cn+d \end{bmatrix} \equiv \binom{a}{c} \begin{bmatrix} b \\ d \end{bmatrix} \pmod{\Phi_n(q)}, \tag{1.1}$$

which is a q-analogue of the well-known Lucas congruence: for any prime p,

$$\binom{ap+b}{cp+d} \equiv \binom{a}{c}\binom{b}{d} \pmod p,$$

where $a, b, c, d \in \mathbb{N}$ with $0 \leq b, d \leq p-1$. Here and throughout the paper, $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q:

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k \leqslant n \\ \gcd(n,k)=1}} (q - \zeta^k),$$

where ζ is an n-th primitive root of unity. The q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{otherwise,} \end{cases}$$

where the *q-shifted factorial* is defined as $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ with $n \in \mathbb{Z}^+$.

On the other hand, for $n \in \mathbb{N}$ and integer j, the trinomial coefficient is the coefficient of x^j in the expansion of $(1+x+x^{-1})^n$. Namely,

$$\left(\!\binom{n}{j}\!\right) = [x^j](1+x+x^{-1})^n,$$

and therefore the relation holds:

$$\left(\!\binom{n}{j}\!\right) = \left(\!\binom{n}{-j}\!\right).$$

The trinomial coefficient has several simple expressions (see[1]), such as

$$\left(\!\binom{n}{j}\!\right) = \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k+j}, \tag{1.2}$$

and

$$\left(\!\binom{n}{j}\!\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2n-2k}{n-j-k}. \tag{1.3}$$

Six different q-analogues of the trinomial coefficients, which play significant roles in hard hexagon model, were introduced by Andrews and Baxter [2]. We list all of them here:

$$\begin{aligned}
\left(\!\binom{n}{j}\!\right)_q^{(B)} &= \sum_{k=0}^{n} q^{k(k+B)} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-k \\ k+j \end{bmatrix}, \\
\tau_0(n,j,q) &= \sum_{k=0}^{n} (-1)^k q^{nk-\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} 2n-2k \\ n-j-k \end{bmatrix}, \\
T_0(n,j,q) &= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \begin{bmatrix} 2n-2k \\ n-j-k \end{bmatrix}, \\
T_1(n,j,q) &= \sum_{k=0}^{n} (-q)^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \begin{bmatrix} 2n-2k \\ n-j-k \end{bmatrix}, \\
t_0(n,j,q) &= \sum_{k=0}^{n} (-1)^k q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \begin{bmatrix} 2n-2k \\ n-j-k \end{bmatrix}, \\
t_1(n,j,q) &= \sum_{k=0}^{n} (-1)^k q^{k^2-k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \begin{bmatrix} 2n-2k \\ n-j-k \end{bmatrix}.
\end{aligned}$$

We point out that $\left(\!\binom{n}{j}\!\right)_q^{(B)}$ with the $B = j$ case has many beautiful and interesting congruence properties, which can be found in [3, 7].

During the past few years, some experts have paid attention to q-supercongruences. We refer the reader to [4, 5, 6, 9, 13, 15, 16] for some of their work. Moreover, some congruences for q-binomial coefficients and q-trinomial coefficients can be found in [3, 7, 8, 10, 14, 17].

Motivated by the work just mentioned, we shall establish six Lucas-like congruences for q-trinomial coefficients.

Theorem 1. *Suppose that $n \geq 2$ and c are integers, $a, b, d \in \mathbb{N}$ with $0 \leq b, d \leq n-1$. There holds*

$$\left(\!\!\binom{an+b}{cn+d}\!\!\right)_q^{(B)} \equiv \left(\!\!\binom{a}{c}\!\!\right)\left(\!\!\binom{b}{d}\!\!\right)_q^{(B)} + \left(\!\!\binom{a}{c+1}\!\!\right)\left(\!\!\binom{b}{d-n}\!\!\right)_q^{(B)} \pmod{\Phi_n(q)}. \tag{1.4}$$

Theorem 2. *Suppose that $n \geq 2$ and c are integers, $a, b, d \in \mathbb{N}$ with $0 \leq b, d \leq n-1$. There holds, modulo $\Phi_n(q)$,*

$$F(an+b, cn+d, q) \equiv \left(\!\!\binom{a}{c}\!\!\right) F(b, d, q) + \left(\!\!\binom{a}{c+1}\!\!\right) F(b, d-n, q),$$

where $F \in \{\tau_0, T_0, T_1, t_0, t_1\}$ represents the other five q-analogues of the trinomial coefficients.

Obviously, taking $q \to 1$ in the above two theorems, we obtain the following congruence for trinomial coefficient.

Corollary 1. *Suppose that a, b, c, d are integers subject to $a \geq 0, 0 \leq b, d \leq p-1$, and p is an odd prime. There holds*

$$\left(\!\!\binom{ap+b}{cp+d}\!\!\right) \equiv \left(\!\!\binom{a}{c}\!\!\right)\left(\!\!\binom{b}{d}\!\!\right) + \left(\!\!\binom{a}{c+1}\!\!\right)\left(\!\!\binom{b}{d-p}\!\!\right) \pmod{p}.$$

Recently, Pan[11] gave a Lucas-type congruence for the q-Delannoy numbers by using a combinatorial interpretation. As serendipitous discoveries, we reprove Pan's curious q-congruence through a different method.

Theorem 3 (Pan[11])**.** *Suppose that $n \geq 2$, $a, b, c, d \in \mathbb{N}$ and $0 \leq b, d \leq n-1$. If n is odd, then*

$$D_q(an+b, cn+d) \equiv D(a, c) D_q(b, d) \pmod{\Phi_n(q)}.$$

If n is even, then

$$D_q(an+b, cn+d) \equiv D_q(b, d) \pmod{\Phi_n(q)}.$$

Here $D_q(n, j)$ and $D(n, j)$ are defined as follows:

$$D_q(n,j) = \sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} j \\ k \end{bmatrix} \begin{bmatrix} n+j-k \\ j \end{bmatrix},$$
$$D(n,j) = \sum_{k=0}^{n} \binom{j}{k} \binom{n+j-k}{j}.$$

It can be easily seen that $D_q(n, j)$ is a q-analogue of $D(n, j)$.

The rest of the paper is arranged as follows. In the next section, we shall give a proof of Theorem 1. The proof of Theorem 2 will be presented in Section 3. In the last section, Theorem 3 will be proven.

2 Proof of Theorem 1

Firstly, we consider the $c \geq 0$ case. If $c > a$, (1.4) holds obviously. Otherwise, expressing the left-hand side of (1.4) by Andrews and Baxter's expression and noting that $\begin{bmatrix} an+b-k \\ cn+d+k \end{bmatrix} = 0$ for any $k > \lfloor \frac{an-cn+b-d}{2} \rfloor$, we get

$$\begin{aligned} \left(\!\!\binom{an+b}{cn+d}\!\!\right)_q^{(B)} &= \sum_{k=0}^{an+b} q^{k(k+B)} \begin{bmatrix} an+b \\ k \end{bmatrix} \begin{bmatrix} an+b-k \\ k+cn+d \end{bmatrix} \\ &= \sum_{k=0}^{\lfloor \frac{an-cn+b-d}{2} \rfloor} q^{k(k+B)} \begin{bmatrix} an+b \\ k \end{bmatrix} \begin{bmatrix} an+b-k \\ k+cn+d \end{bmatrix} \\ &= \Sigma_1 + \Sigma_2, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \Sigma_1 &= \sum_{k=0}^{\lfloor \frac{a-c}{2} \rfloor} q^{kn(kn+B)} \begin{bmatrix} an+b \\ kn \end{bmatrix} \begin{bmatrix} an+b-kn \\ kn+cn+d \end{bmatrix}, \\ \Sigma_2 &= \sum_{j=0}^{\lfloor \frac{a-c}{2} \rfloor} \sum_{k=jn+1}^{jn+n-1} q^{k(k+B)} \begin{bmatrix} an+b \\ k \end{bmatrix} \begin{bmatrix} an+b-k \\ k+cn+d \end{bmatrix}. \end{aligned}$$

Since $1 \leq b, d \leq n-1$, congruence (1.1) takes effect here. Applying congruence (1.1) to each of the q-binomial coefficients in the summand of Σ_1 and keeping in mind that $q^n \equiv 1 \pmod{\Phi_n(q)}$, we obtain that

$$\Sigma_1 \equiv \sum_{k=0}^{\lfloor \frac{a-c}{2} \rfloor} \binom{a}{k} \begin{bmatrix} b \\ 0 \end{bmatrix} \binom{a-k}{k+c} \begin{bmatrix} b \\ d \end{bmatrix} \equiv \left(\!\!\binom{a}{c}\!\!\right) \begin{bmatrix} b \\ d \end{bmatrix} \pmod{\Phi_n(q)}, \tag{2.2}$$

where we have utilized the expression of trinomial coefficient (1.2) in the last relation.

With the help of congruence (1.1) again, there holds

$$\begin{aligned} \Sigma_2 &= \sum_{j=0}^{\lfloor \frac{a-c}{2} \rfloor} \sum_{k=1}^{n-1} q^{(k+jn)(k+jn+B)} \begin{bmatrix} an+b \\ jn+k \end{bmatrix} \begin{bmatrix} an+b-k-jn \\ k+jn+cn+d \end{bmatrix} \\ &\equiv \sum_{j=0}^{\lfloor \frac{a-c}{2} \rfloor} \sum_{k=1}^{n-1-d} q^{k(k+B)} \binom{a}{j} \begin{bmatrix} b \\ k \end{bmatrix} \binom{a-j}{j+c} \begin{bmatrix} b-k \\ k+d \end{bmatrix} \\ &\quad + \sum_{j=0}^{\lfloor \frac{a-c}{2} \rfloor} \sum_{k=n-d}^{n-1} q^{k(k+B)} \binom{a}{j} \begin{bmatrix} b \\ k \end{bmatrix} \binom{a-j}{j+c+1} \begin{bmatrix} b-k \\ k+d-n \end{bmatrix} \\ &\equiv \left(\!\!\binom{a}{c}\!\!\right) \left(\!\!\binom{b}{d}\!\!\right)_q^{(B)} - \left(\!\!\binom{a}{c}\!\!\right) \begin{bmatrix} b \\ d \end{bmatrix} + \left(\!\!\binom{a}{c+1}\!\!\right) \left(\!\!\binom{b}{d-n}\!\!\right)_q^{(B)} \pmod{\Phi_n(q)}. \end{aligned} \tag{2.3}$$

By substituting the results (2.2) and (2.3) into the right-hand side of (2.1), we get the desired result: for $n \geq 2$ and c are non-negative integers, $a, b, d \in \mathbb{N}$ with $0 \leq b, d \leq n-1$, there holds

$$\left(\!\!\binom{an+b}{cn+d}\!\!\right)_q^{(B)} \equiv \left(\!\!\binom{a}{c}\!\!\right)\left(\!\!\binom{b}{d}\!\!\right)_q^{(B)} + \left(\!\!\binom{a}{c+1}\!\!\right)\left(\!\!\binom{b}{d-n}\!\!\right)_q^{(B)} \pmod{\Phi_n(q)}. \tag{2.4}$$

Now suppose that $c < 0, 1 \leq d \leq n-1$. Then utilizing (2.4) and the following easily-proved relation,

$$\left(\!\!\binom{n}{-j}\!\!\right)_q^{(B)} = \left(\!\!\binom{n}{j}\!\!\right)_q^{(B+2j)} q^{j(j+B)},$$

we obtain

$$\begin{aligned}
\left(\!\!\binom{an+b}{cn+d}\!\!\right)_q^{(B)} &\equiv \left(\!\!\binom{an+b}{-cn-n+n-d}\!\!\right)_q^{(B-2cn-2d)} q^{(cn+d)(cn+d-B)} \\
&\equiv \left(\!\!\binom{a}{-c-1}\!\!\right)\left(\!\!\binom{b}{n-d}\!\!\right)_q^{(B-2cn-2d)} q^{(cn+d)(cn+d-B)} \\
&\quad + \left(\!\!\binom{a}{-c}\!\!\right)\left(\!\!\binom{b}{-d}\!\!\right)_q^{(B-2cn-2d)} q^{(cn+d)(cn+d-B)} \\
&\equiv \left(\!\!\binom{a}{c}\!\!\right)\left(\!\!\binom{b}{d}\!\!\right)_q^{(B-2cn)} q^{(cn+d)(cn+d-B)+d(B-2cn-d)} \\
&\quad + \left(\!\!\binom{a}{c+1}\!\!\right)\left(\!\!\binom{b}{d-n}\!\!\right)_q^{(B-2cn-2n)} q^{(cn+d)(cn+d-B)+(d-n)(B-n-2cn-d)} \\
&\equiv \left(\!\!\binom{a}{c}\!\!\right)\left(\!\!\binom{b}{d}\!\!\right)_q^{(B)} + \left(\!\!\binom{a}{c+1}\!\!\right)\left(\!\!\binom{b}{d-n}\!\!\right)_q^{(B)} \pmod{\Phi_n(q)}.
\end{aligned}$$

And it can be easily seen that this relation holds for $d = 0$ as well. Hence we finish the whole proof.

3 Proof of Theorem 2

The proof of Theorem 2 is more complicated than that of Theorem 1, since different ranges of b, d, k mean different congruence results. Therefore, we have to deal with the binomial coefficients in the summand case by case.

Let's start with the $F = \tau_0$ case. We first split the summation of the following q-trinomial coefficients into two parts as

$$\begin{aligned}
\tau_0(an+b, cn+d, q) &= \sum_{k=0}^{an-cn+b-d} (-1)^k q^{(an+b)k-\binom{k}{2}} \begin{bmatrix} an+b \\ k \end{bmatrix} \begin{bmatrix} 2an+2b-2k \\ an-cn+b-d-k \end{bmatrix} \\
&= \Sigma_1 + \Sigma_2,
\end{aligned} \tag{3.1}$$

where

$$\Sigma_1 = \sum_{k=0}^{a-c} (-1)^{kn} q^{(an+b)kn-\binom{kn}{2}} \begin{bmatrix} an+b \\ kn \end{bmatrix} \begin{bmatrix} 2an+2b-2kn \\ an-cn+b-d-kn \end{bmatrix},$$

$$\Sigma_2 = \sum_{j=0}^{a-c} \sum_{k=1}^{n-1} (-1)^{k+jn} q^{(an+b)(k+jn)-\binom{k+jn}{2}} \begin{bmatrix} an+b \\ k+jn \end{bmatrix} \begin{bmatrix} 2an+2b-2k-2jn \\ an-cn+b-d-k-jn \end{bmatrix}.$$

Now we shall discuss the different cases of b and d one by one. The first case is $b < d$ and $b \le (n-1)/2$. In this case, we have, modulo $\Phi_n(q)$,

$$\Sigma_1 \equiv \sum_{k=0}^{a-c} (-1)^k \binom{a}{k} \binom{2a-2k}{a-c-k-1} \begin{bmatrix} 2b \\ b-d+n \end{bmatrix}, \tag{3.2}$$

$$\Sigma_2 \equiv \sum_{j=0}^{a-c} \sum_{k=1}^{n-1} (-1)^{k+j} q^{bk-\binom{k}{2}} \binom{a}{j} \binom{2a-2j}{a-c-j-1} \begin{bmatrix} b \\ k \end{bmatrix} \begin{bmatrix} 2b-2k \\ b-d-k+n \end{bmatrix}, \tag{3.3}$$

where we have utilized congruence (1.1) repeatedly with $0 \le 2b, b-d+n, 2b-2k, b-d-k+n \le n-1$.

Combining (3.1), (3.2) with (3.3) together, and noticing the trinomial coefficient's expression (1.3), we are led to

$$\tau_0(an+b, cn+d, q) \equiv \left(\!\!\binom{a}{c+1}\!\!\right) \tau_0(b, d-n, q) \pmod{\Phi_n(q)}.$$

The second case is $b < d$ and $b \ge (n+1)/2$. Then, modulo $\Phi_n(q)$,

$$\begin{aligned} \Sigma_1 &\equiv \sum_{k=0}^{a-c} (-1)^k \binom{a}{k} \binom{2a-2k+1}{a-c-k-1} \begin{bmatrix} 2b-n \\ b-d+n \end{bmatrix}, \\ \Sigma_2 &\equiv \sum_{j=0}^{a-c} \sum_{k=1}^{b-(n+1)/2} (-1)^{k+j} q^{bk-\binom{k}{2}} \binom{a}{j} \binom{2a-2j+1}{a-c-j-1} \begin{bmatrix} b \\ k \end{bmatrix} \begin{bmatrix} 2b-2k-n \\ b-d-k+n \end{bmatrix} \\ &\quad + \sum_{j=0}^{a-c} \sum_{k=b-(n-1)/2}^{n-1} (-1)^{k+j} q^{bk-\binom{k}{2}} \binom{a}{j} \binom{2a-2j}{a-c-j-1} \begin{bmatrix} b \\ k \end{bmatrix} \begin{bmatrix} 2b-2k \\ b-d-k+n \end{bmatrix}. \end{aligned}$$

Noticing that $\left[\begin{smallmatrix} 2b-2k-n \\ b-d-k+n \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} 2b-2k \\ b-d-k+n \end{smallmatrix}\right] \equiv 0 \pmod{\Phi_n(q)}$ for $k \le b-(n+1)/2$, we obtain

$$\Sigma_1 + \Sigma_2 \equiv \left(\!\!\binom{a}{c+1}\!\!\right) \tau_0(b, d-n, q) \pmod{\Phi_n(q)}.$$

The third case is $b \geq d$ and $b \leq (n-1)/2$. Through the same path, we get

$$\begin{aligned}
\Sigma_1 &\equiv \sum_{k=0}^{a-c}(-1)^k \binom{a}{k}\binom{2a-2k}{a-c-k}\begin{bmatrix}2b\\ b-d\end{bmatrix} \pmod{\Phi_n(q)},\\
\Sigma_2 &\equiv \sum_{j=0}^{a-c}\sum_{k=1}^{b-d}(-1)^{k+j}q^{bk-\binom{k}{2}}\binom{a}{j}\binom{2a-2j}{a-c-j}\begin{bmatrix}b\\ k\end{bmatrix}\begin{bmatrix}2b-2k\\ b-d-k\end{bmatrix}\\
&\quad+\sum_{j=0}^{a-c}\sum_{k=b-d+1}^{n-1}(-1)^{k+j}q^{bk-\binom{k}{2}}\binom{a}{j}\binom{2a-2j}{a-c-j-1}\begin{bmatrix}b\\ k\end{bmatrix}\begin{bmatrix}2b-2k\\ b-d-k+n\end{bmatrix} \pmod{\Phi_n(q)}.
\end{aligned}$$

Observing that $\begin{bmatrix}2b-2k\\ b-d-k+n\end{bmatrix}$ is always equal to 0 for $b-d+1 \leq k \leq n-1$, we have

$$\tau_0(an+b, cn+d, q) \equiv \left(\binom{a}{c}\right)\tau_0(b,d,q) \pmod{\Phi_n(q)}.$$

The next case is $b \geq d$, $b \geq (n+1)/2$ and $d \leq (n-1)/2$. In such condition, we have: modulo $\Phi_n(q)$,

$$\begin{aligned}
\Sigma_1 &\equiv \sum_{k=0}^{a-c}(-1)^k \binom{a}{k}\binom{2a-2k+1}{a-c-k}\begin{bmatrix}2b-n\\ b-d\end{bmatrix},\\
\Sigma_2 &\equiv \sum_{j=0}^{a-c}\sum_{k=1}^{b-(n+1)/2}(-1)^{k+j}q^{bk-\binom{k}{2}}\binom{a}{j}\binom{2a-2j+1}{a-c-j}\begin{bmatrix}b\\ k\end{bmatrix}\begin{bmatrix}2b-2k-n\\ b-d-k\end{bmatrix}\\
&\quad+\sum_{j=0}^{a-c}\sum_{k=b-(n-1)/2}^{b-d}(-1)^{k+j}q^{bk-\binom{k}{2}}\binom{a}{j}\binom{2a-2j}{a-c-j}\begin{bmatrix}b\\ k\end{bmatrix}\begin{bmatrix}2b-2k\\ b-d-k\end{bmatrix}\\
&\quad+\sum_{j=0}^{a-c}\sum_{k=b-d+1}^{n-1}(-1)^{k+j}q^{bk-\binom{k}{2}}\binom{a}{j}\binom{2a-2j}{a-c-j-1}\begin{bmatrix}b\\ k\end{bmatrix}\begin{bmatrix}2b-2k\\ b-d-k+n\end{bmatrix}.
\end{aligned}$$

The third summation of Σ_2 could be cancelled since $\begin{bmatrix}2b-2k\\ b-d-k+n\end{bmatrix} = 0$ for $k > b+d-n$, and $b+d-n \leq b-d+1$ always holds in this case.

Due to the relation

$$\binom{2a-2j+1}{a-c-j} = \binom{2a-2j}{a-c-j} + \binom{2a-2j}{a-c-j-1}, \tag{3.4}$$

and the fact that, for $0 \leq k \leq b-(n+1)/2$,

$$\begin{bmatrix}2b-2k-n\\ b-d-k\end{bmatrix} \equiv \begin{bmatrix}2b-2k\\ b-d-k\end{bmatrix} \equiv \begin{bmatrix}2b-2k\\ b-d-k+n\end{bmatrix} \pmod{\Phi_n(q)}, \tag{3.5}$$

we can simplify $\Sigma_1 + \Sigma_2$ as follows:

$$\begin{aligned}\Sigma_1 + \Sigma_2 \equiv& \left(\!\!\binom{a}{c}\!\!\right) \sum_{k=0}^{b-(n+1)/2} (-1)^k q^{bk-\binom{k}{2}} \begin{bmatrix} b \\ k \end{bmatrix} \begin{bmatrix} 2b-2k-n \\ b-d-k \end{bmatrix} \\ &+ \left(\!\!\binom{a}{c+1}\!\!\right) \sum_{k=0}^{b-(n+1)/2} (-1)^k q^{bk-\binom{k}{2}} \begin{bmatrix} b \\ k \end{bmatrix} \begin{bmatrix} 2b-2k-n \\ b-d-k \end{bmatrix} \\ &+ \left(\!\!\binom{a}{c}\!\!\right) \tau_0(b,d,q) - \left(\!\!\binom{a}{c}\!\!\right) \sum_{k=0}^{b-(n+1)/2} (-1)^k q^{bk-\binom{k}{2}} \begin{bmatrix} b \\ k \end{bmatrix} \begin{bmatrix} 2b-2k \\ b-d-k \end{bmatrix} \\ \equiv& \left(\!\!\binom{a}{c}\!\!\right) \tau_0(b,d,q) + \left(\!\!\binom{a}{c+1}\!\!\right) \tau_0(b,d-n,q) \pmod{\Phi_n(q)}.\end{aligned}$$

The last case is $b \geq d$, $b \geq (n+1)/2$ and $d \geq (n+1)/2$. In this case, modulo $\Phi_n(q)$,

$$\begin{aligned}\Sigma_1 \equiv& \sum_{k=0}^{a-c} (-1)^k \binom{a}{k} \binom{2a-2k+1}{a-c-k} \begin{bmatrix} 2b-n \\ b-d \end{bmatrix}, \\ \Sigma_2 \equiv& \sum_{j=0}^{a-c} \sum_{k=1}^{b-d} (-1)^{k+j} q^{bk-\binom{k}{2}} \binom{a}{j} \binom{2a-2j+1}{a-c-j} \begin{bmatrix} b \\ k \end{bmatrix} \begin{bmatrix} 2b-2k-n \\ b-d-k \end{bmatrix} \\ &+ \sum_{j=0}^{a-c} \sum_{k=b-d+1}^{b-(n+1)/2} (-1)^{k+j} q^{bk-\binom{k}{2}} \binom{a}{j} \binom{2a-2j}{a-c-j} \begin{bmatrix} b \\ k \end{bmatrix} \begin{bmatrix} 2b-2k-n \\ b-d-k+n \end{bmatrix} \\ &+ \sum_{j=0}^{a-c} \sum_{k=b-(n-1)/2}^{n-1} (-1)^{k+j} q^{bk-\binom{k}{2}} \binom{a}{j} \binom{2a-2j}{a-c-j-1} \begin{bmatrix} b \\ k \end{bmatrix} \begin{bmatrix} 2b-2k \\ b-d-k+n \end{bmatrix}.\end{aligned}$$

Using relation (3.4) and (3.5) again and following the similar path in the fourth case, we are access to the conclusion

$$\tau_0(an+b, cn+d, q) \equiv \left(\!\!\binom{a}{c}\!\!\right) \tau_0(b,d,q) + \left(\!\!\binom{a}{c+1}\!\!\right) \tau_0(b,d-n,q) \pmod{\Phi_n(q)}.$$

All of these five cases satisfy the congruence in Theorem 2 as desired.

The result in $c < 0$ case can be deduced easily due to the relation $\tau_0(n,j,q) = \tau_0(n,-j,q)$, then the proof is just like what we have done in the proof of Theorem 1.

Taking $q \to q^2$ in (1.1), we get

$$\begin{bmatrix} an+b \\ cn+d \end{bmatrix}_{q^2} \equiv \binom{a}{c} \begin{bmatrix} b \\ d \end{bmatrix}_{q^2} \pmod{\Phi_n(q^2)}.$$

Then there holds

$$\begin{bmatrix} an+b \\ cn+d \end{bmatrix}_{q^2} \equiv \binom{a}{c} \begin{bmatrix} b \\ d \end{bmatrix}_{q^2} \pmod{\Phi_n(q)}, \tag{3.6}$$

since $\Phi_n(q)$ divides $\Phi_n(q^2)$.

Utilizing (3.6) instead of (1.1) if it is necessary, we find that the processes of proving the congruence about $F \in \{T_0, T_1, t_0, t_1\}$ are totally the same as that of proving the $F = \tau_0$ case, and therefore we omit their proofs here.

4 Proof of Theorem 3

We split the summation of the following Delannoy number into two parts as usual. Firstly, we consider the situation where n is an odd integer.

$$D_q(an+b, cn+d) = \sum_{k=0}^{an+b} q^{\binom{k+1}{2}} \begin{bmatrix} cn+d \\ k \end{bmatrix} \begin{bmatrix} an+cn+b+d-k \\ cn+d \end{bmatrix} = \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \sum_{k=0}^{a} q^{\binom{kn+1}{2}} \begin{bmatrix} cn+d \\ kn \end{bmatrix} \begin{bmatrix} an+cn+b+d-kn \\ cn+d \end{bmatrix},$$

$$\Sigma_2 = \sum_{j=0}^{a} \sum_{k=1}^{n-1} q^{\binom{k+jn+1}{2}} \begin{bmatrix} cn+d \\ k+jn \end{bmatrix} \begin{bmatrix} an+cn+b+d-k-jn \\ cn+d \end{bmatrix}.$$

This time, we need to deal with two different cases. The first case is $b+d \le n-1$. In this case, modulo $\Phi_n(q)$,

$$\Sigma_1 \equiv \sum_{k=0}^{a} \binom{c}{k} \binom{a+c-k}{c} \begin{bmatrix} b+d \\ d \end{bmatrix},$$

$$\Sigma_2 \equiv \sum_{j=0}^{a} \sum_{k=1}^{n-1} q^{\binom{k+1}{2}} \binom{c}{j} \binom{a+c-j}{c} \begin{bmatrix} d \\ k \end{bmatrix} \begin{bmatrix} b+d-k \\ d \end{bmatrix}.$$

Then immediately we get the result

$$\Sigma_1 + \Sigma_2 \equiv D(a,c) D_q(b,d) \pmod{\Phi_n(q)}.$$

The other case is $b+d \ge n$. Then, modulo $\Phi_n(q)$,

$$\Sigma_1 \equiv \sum_{k=0}^{a} \binom{c}{k} \binom{a+c-k+1}{c} \begin{bmatrix} b+d-n \\ d \end{bmatrix},$$

$$\begin{aligned}\Sigma_2 \equiv &\sum_{j=0}^{a} \sum_{k=1}^{b+d-n} q^{\binom{k+1}{2}} \binom{c}{j} \binom{a+c-j+1}{c} \begin{bmatrix} d \\ k \end{bmatrix} \begin{bmatrix} b+d-k-n \\ d \end{bmatrix} \\ &+ \sum_{j=0}^{a} \sum_{k=b+d-n+1}^{n-1} q^{\binom{k+1}{2}} \binom{c}{j} \binom{a+c-j}{c} \begin{bmatrix} d \\ k \end{bmatrix} \begin{bmatrix} b+d-k \\ d \end{bmatrix}.\end{aligned}$$

It is not difficult to verify that for $0 \le k \le b+d-n$

$$\begin{bmatrix} b+d-k-n \\ d \end{bmatrix} \equiv \begin{bmatrix} b+d-k \\ d \end{bmatrix} \equiv 0 \pmod{\Phi_n(q)}.$$

As a result,

$$\begin{aligned}\Sigma_1 + \Sigma_2 &\equiv \sum_{j=0}^{a} \sum_{k=0}^{n-1} q^{\binom{k+1}{2}} \binom{c}{j} \binom{a+c-j}{c} \begin{bmatrix} d \\ k \end{bmatrix} \begin{bmatrix} b+d-k \\ d \end{bmatrix} \\ &\equiv D(a,c) D_q(b,d) \pmod{\Phi_n(q)}.\end{aligned}$$

Now we consider the other situation where n is an even integer. Noticing that $q^{n/2} \equiv -1 \pmod{\Phi_n(q)}$, we have, modulo $\Phi_n(q)$,

$$q^{\binom{kn+1}{2}} \equiv (-1)^k \qquad \text{and} \qquad q^{\binom{jn+k+1}{2}} \equiv (-1)^j q^{\binom{k+1}{2}}.$$

Therefore,

$$\Sigma_1 + \Sigma_2 \equiv \sum_{j=0}^{a}\sum_{k=0}^{n-1}(-1)^j q^{\binom{k+1}{2}}\binom{c}{j}\binom{a+c-j}{c}\begin{bmatrix} d \\ k \end{bmatrix}\begin{bmatrix} b+d-k \\ d \end{bmatrix} \pmod{\Phi_n(q)}.$$

In order to finish proving Theorem 3, it only remains to show

$$\sum_{j=0}^{a}(-1)^j\binom{c}{j}\binom{a+c-j}{c} = 1.$$

In fact,

$$\sum_{j=0}^{a}(-1)^j\binom{c}{j}\binom{a+c-j}{c} = \frac{(1)_{a+c}}{(1)_c(1)_a}\,{}_2F_1\left[\begin{matrix} -a,-c \\ -a-c \end{matrix};1\right] = \frac{(1)_{a+c}}{(1)_c(1)_a}\frac{(-a)_a}{(-a-c)_a} = 1,$$

where we have utilized the Chu-Vandermonde formula

$${}_2F_1\left[\begin{matrix} -n,a \\ c \end{matrix};1\right] = \frac{(c-a)_n}{(c)_n}.$$

Now the proof is completed.

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(1) Department of Mathematics, Shanghai University, Shanghai 200444, P. R. China
E-mail: `chenyf576@shu.edu.cn`

(2) Department of Mathematics, Shanghai University, Shanghai 200444, P. R. China
and
Newtouch Center for Mathematics of Shanghai University, Shanghai 200444, P. R. China
E-mail: `xiaoxiawang@shu.edu.cn` (Corresponding author)