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Two modified inertial projection methods for solving quasimonotone variational inequalities

by

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Abstract

In this paper, we provide two inertial projection methods with a novel nonmonotonic adaptive step size for solving variational inequalities governed by quasimonotone and Lipschitz continuous operators in real Hilbert spaces. Compared with the general subgradient extragradient method, our algorithms use a different half-space. Under some suitable conditions, we obtain the weak convergence theorem of the first modified inertial projection algorithm and the strong convergence theorem of the second modified viscosity-type inertial projection algorithm. Moreover, several numerical results are given to illustrate the effectiveness and competitiveness of our proposed methods.

Key Words: Variational inequality, inertial technique, quasimonotone, Hilbert space.
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1 Introduction

Throughout this paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. It is well known that the classical variational inequality problem (VIP, for short) is defined as: find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \ \forall y \in C, \tag{1.1}$$

where C is a nonempty, closed, and convex subset of H and $A : H \to H$ is a continuous mapping. The solution set of VIP is denoted by S.

The variational inequality problem is a key problem in nonlinear analysis, which has been widely used in nonlinear programming, network equilibrium problems and complementarity problems (see, for example, [7, 12, 19] and the references therein). So far, a number of iterative algorithms for VIP have been proposed (see, for example, [1,4,5,15,17,18,27,33,35]), we mainly focus on the projection-type methods.

One of the well-known methods for solving problem (1.1) in Euclidean spaces is the extragradient method (EGM), which was presented by Korpelevich in [13]. In recent years, the extragradient method has been extended to infinite spaces in various ways (see, for example, [2, 9, 23, 24, 30]). The EGM needs to calculate two projection values onto the feasible set per iteration, however, computing projection onto a general closed and convex set might be difficult. Two famous methods were proposed to overcome the major drawback of this method. The first algorithm is the subgradient extragradient method (SEGM), which was introduced by Censor et al. [3]. This method uses a projection onto a specific half-space in place of the second projection onto C in the EGM. The second method is the Tseng extragradient method (TEGM), which was presented by Tseng [29]. This method uses an explicit formula replaces the second projection onto C in the EGM. Moreover, the weak convergence results of the EGM, the SEGM and the TEGM were obtained when the cost operator is pseudomonotone and Lipschitz continuous.

Recently, many authors are committed to investigating algorithms with inertial extrapolation terms, which can be used to speed up the convergence of iterative methods (see, for example [6,8,14,25,26,31]). Inspired by Censor et al. [3], Yang [32] proposed a self-adaptive inertial subgradient extragradient algorithm. It is of the form: Take $\lambda_1 > 0, x_0, x_1 \in H$ and $0 < \mu < \mu_0 < 1$,

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}) \\ y_n = P_C(w_n - \lambda_n F w_n) \\ T_n := \{w \in H : \langle w_n - \lambda_n F w_n - y_n, w - y_n\rangle \leq 0\} \\ x_{n+1} = P_{T_n}(w_n - \lambda_n F y_n), \end{cases} \tag{1.2}$$

where

$$\lambda_{n+1} = \begin{cases} \min\Big\{\dfrac{\mu(\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2\langle F w_n - F y_n, x_{n+1} - y_n\rangle}, \lambda_n\Big\}, & if \langle F w_n - F y_n, x_{n+1} - y_n\rangle > 0, \\ \lambda_n, & otherwise, \end{cases}$$

and $0 \leq \alpha_n \leq \alpha_{n+1} \leq \alpha$, $\alpha < -2\theta_0 - 1 + \sqrt{8\theta_0 + 1}/2(1 - \theta_0)$, $\theta_0 = \frac{1-\mu_0}{2}$. $\{x_n\}$ converges weakly to a solution of VIP when F is pseudomonotone.

Thong et al. [28] proposed a novel projection method for solving pseudomonotone variational inequality problems and gave several numerical experiments to show that this algorithm converges faster than the EGM and the SEGM. It is of the form: Take $\lambda_0 > 0, u_1 \in H$ and $\mu \in (0, 1)$, let $\{\alpha_n\}$ be a nonnegative real sequence such that $\sum\limits_{n=1}^{\infty} \alpha_n < +\infty$,

$$\begin{cases} v_n = P_C(u_n - \lambda_n F u_n) \\ u_{n+1} = P_{T_n}(u_n) \\ T_n := \{x \in H : h_n(x) \leq 0\} \end{cases} \tag{1.3}$$

where

$$h_n(x) := \langle u_n - v_n - \lambda_n(F u_n - F v_n), x - v_n\rangle,$$

$$\lambda_{n+1} = \begin{cases} \min\Big\{\dfrac{\mu\|u_n - v_n\|}{\|F u_n - F v_n\|}, \lambda_n + \alpha_n\Big\}, & if F u_n - F v_n \neq 0, \\ \lambda_n + \alpha_n, & otherwise. \end{cases}$$

The sequence $\{x_n\}$ generated by algorithm (1.3) converges weakly to a solution of VIP when the operator F is pseudomonotone and Lipschitz continuous.

Motivated by algorithm (1.2) and (1.3), we propose two new inertial projection algorithms for solving quasimonotone variational inequalities in this paper. Moreover, we present some numerical experiments to show that our algorithm converges faster than algorithm (1.2), (1.3) and Algorithm 1 in [22].

Our contributions:

• We introduce two projection methods with single inertial extrapolation step to solve the variational inequality problem in real Hilbert spaces, our methods accelerate the convergence rates of the methods in [22, 28, 32] effectively. Our algorithm has the following advantages: (1) both of our algorithms have an inertia term that speeds up the rate of convergence; (2) we obtain weak convergence result and strong convergence result under a weaker operator condition in our algorithms (A is quasi-monotone rather than pseudo-monotone); (3) we take a modified generalized non-monotonic step size, which accelerates the convergence rate effectively. (Figure 1-4 in Numerical experiments show the superiority of our algorithms' step size)

• Under some suitable conditions, we obtain the weak convergence theorem of the first modified inertial projection algorithm 1 and the strong convergence theorem of the second modified viscosity-type inertial projection algorithm 2.

• We give numerical simulations to show that our proposed methods is more efficient and faster than the related methods.

Our paper is organized as follows: Several definitions and lemmas are given in Sect. 2. In Sect. 3, we present our method and analyse the weak convergence of our method. We give numerical experiments to illustrate the feasibility of our methods in Sect. 4.

2 Preliminaries

Definition 2.1. The operator $A : H \to H$ is said to be

(i) L-Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \ \forall x, y \in H.$$

(ii) ϱ-strongly monotone, if there exists a constant $\varrho > 0$ such that

$$\langle Ay - Ax, y - x\rangle \geq \varrho\|y - x\|^2, \ \forall x, y \in H.$$

(iii) monotone, if

$$\langle Ay - Ax, y - x\rangle \geq 0, \ \forall x, y \in H.$$

(iv) η-strongly pseudomonotone, if there exists a constant $\eta > 0$ such that

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq \eta\|y - x\|^2, \ \forall x, y \in H.$$

(v) pseudomonotone, if

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \ \forall x, y \in H.$$

(vi) quasimonotone, if

$$\langle Ax, y - x\rangle > 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \ \forall x, y \in H.$$

Clearly, $(ii) \Rightarrow (iii) \Rightarrow (v) \Rightarrow (vi)$ and $(ii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi)$, but the converses are not always true.
The dual variational inequality problem (shortly, DVIP) is defined as: find $x \in C$ such that

$$\langle Ay, y - x\rangle \geq 0, \forall y \in C. \tag{2.1}$$

The solution set of DVIP is denoted by S_D. When A is quasimonotone, we have S_D is a closed and convex subset of C. Furthermore, since C is convex and A is continuous, we have $S_D \subset S$.

Lemma 2.2. *([34]) Let C be a nonempty closed and convex subset of H. If either*
(i) A is pseudomonotone on C and $S \neq \emptyset$,
(ii) A is the gradient of G, where G is a differential quasiconvex function on an open set $K \supset C$
and attains its global minimum on C,
(iii) A is quasimonotone on C, $A \neq 0$ on C and C is bounded,
(iv) A is quasimonotone on C, $A \neq 0$ on C and there exists a positive number r such that, for every $v \in C$ with $\|v\| \geq r$, there exists $y \in C$ such that $\|y\| \leq r$ and $\langle Av, y - v\rangle \leq 0$,
(v) A is quasimonotone on C, $intC$ is nonempty and there exists $v^* \in S$ such that $Av^* \neq 0$.
Then, S_D is nonempty.

Lemma 2.3. *The following statements hold in H:*
(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \ \forall \ x, y \in H$.
(ii)$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \ \forall \ x, y \in H$.
(iii)$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2, \ \forall \ x, y \in H, \ \lambda \in R$.

Lemma 2.4. *Let C be a nonempty closed and convex subset of H and P_C be the metric projection from H onto C. Then for any $x, y \in H$ and $z \in C$, the following hold:*
(i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$.
(ii)$\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2$.

Lemma 2.5. *For any $x \in H$ and $z \in C$, then $z = P_C(x)$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \forall y \in C.$$

Lemma 2.6. *Let H be a real Hilbert space and let h be a real-valued function on H. Define $K := \{x \in H : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on H with modulus $\theta > 0$, then*

$$dist(x, K) \geq \theta^{-1} \max\{h(x), 0\}, \ \forall x \in H,$$

where $dist(x, K)$ denotes the distance function from x to K.

Proof. The proof is similar to Lemma 2.3 in [11]. □

Lemma 2.7. *([16], Lemma 2.2) Let $\{\phi_n\}, \{\delta_n\}$ and $\{\theta_n\}$ be sequences in $[0, +\infty)$ such that*

$$\phi_{n+1} \leq \phi_n + \theta_n(\phi_n - \phi_{n-1}) + \delta_n, \forall n \geq 1, \sum_{n=1}^{\infty} \delta_n < +\infty,$$

and exists a real number θ with $0 \le \theta_n \le \theta < 1$ for all $n \in N$. Then the following assertions hold:

(i) $\sum_{n=1}^{\infty} [\phi_n - \phi_{n-1}]_+ < +\infty$ *where* $[t]_+ = \max\{t, 0\}$ *for any* $t \in R$;
(ii) there exists $\phi^* \in [0, +\infty)$ *such that* $\lim_{n\to\infty} \phi_n = \phi^*$.

Lemma 2.8. *([20]) Let C be a nonempty subset of H and let $\{x_n\}$ be a sequence in H such that the following two conditions:*
(i) for each $x \in C$, the limit of sequence $\{\|x_n - x\|\}$ exists;
(ii) any weak cluster point of sequence $\{x_n\}$ is in C.
Then there exists $x^* \in C$ such that $\{x_n\}$ converges weakly to x^*.

Lemma 2.9. *([21]) Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n b_n, \ \forall n \ge 1,$$

if $\limsup_{k\to\infty} b_{n_k} \le 0$ *for every subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying* $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0$, *then* $\lim_{n\to\infty} a_n = 0$.

3 Convergence analysis

In this section, we show that the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a point in $S_D \subset S$ and the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to a point in $S_D \subset S$ under suitable conditions.

3.1 Modified inertial projection algorithm

Condition 1 The feasible set C is a nonempty, closed and convex subset of H;
Condition 2 The operator $A : H \to H$ is quasimonotone and L-Lipschitz continuous;
Condition 3 The operator $A : H \to H$ satisfies the following condition:

$$\text{if } \{x_n\} \subset H,\ x_n \rightharpoonup v^* \text{ and } \liminf_{n\to\infty} \|Ax_n\| = 0, \text{ then } Av^* = 0;$$

Condition 4 $S_D \neq \emptyset$;

Algorithm 1 Modified inertial projection algorithm

Iterative step:
1. Take the parameters $\mu \in (0,1)$ and $\lambda_1 > 0$. Choose $\{\theta_n\}$ such that $0 \le \theta_n \le \theta_{n+1} \le \frac{1}{4}$, $\{a_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} a_n < +\infty$ and $\{q_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} q_n = 1$. Let

$x_0, x_1 \in H$ be given starting points. Set n:=1.
2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}),$$
$$y_n = P_C(w_n - \lambda_n A w_n).$$

If $w_n = y_n$ or $Aw_n = 0$, STOP. Otherwize,
3. Compute

$$x_{n+1} = P_{T_n}(w_n),$$

where

$$T_n := \{w \in H : h_n(w) \leq 0\}$$

and

$$h_n(w) := \langle w_n - y_n - \lambda_n(Aw_n - Ay_n), w - y_n\rangle.$$

Update

$$\lambda_{n+1} = \begin{cases} \min\Big\{\frac{\mu q_n\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n + a_n\Big\}, & if\, Aw_n - Ay_n \neq 0, \\ \lambda_n + a_n, & otherwise. \end{cases} \tag{3.1}$$

4. Set $n \leftarrow n+1$, and go to 2.

Lemma 3.1. *Suppose that Condition 2 holds, then the sequence $\{\lambda_n\}$ generated by (3.1) is well defined and $\lim\limits_{n\to\infty} \lambda_n = \lambda$ and $\lambda \in [\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + \sum\limits_{n=1}^{\infty} a_n]$. Moreover, we also have*

$$\|Aw_n - Ay_n\| \leq \frac{\mu q_n}{\lambda_{n+1}}\|w_n - y_n\|. \tag{3.2}$$

Proof. The proof is similar to Lemma 3.3 in [15]. ∎

Lemma 3.2. *Assume that Condition 1-4 hold and $\{x_n\}$ is a sequence generated by Algorithm 1. Then $\{x_n\}$ is bounded and $\lim\limits_{n\to\infty}\|x_n - x^*\|$ exists, where $x^* \in S_D$.*

Proof. From Lemma 2.4, we have

$$\|x_{n+1} - x^*\|^2 = \|P_{T_n}(w_n) - x^*\|^2 \leq \|w_n - x^*\|^2 - \|x_{n+1} - w_n\|^2. \tag{3.3}$$

From Lemma 2.3, we have

$$\begin{aligned} \|w_n - x^*\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\|^2 \\ &= \|(1+\theta_n)(x_n - x^*) - \theta_n(x_{n-1} - x^*)\|^2 \\ &= (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2, \end{aligned} \tag{3.4}$$

$$\|x_{n+1} - w_n\|^2 = \|x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\|^2$$

$$
\begin{aligned}
&= \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 - 2\theta_n \langle x_{n+1} - x_n, x_n - x_{n-1} \rangle \\
&\geq \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 - 2\theta_n \|x_{n+1} - x_n\| \, \|x_n - x_{n-1}\| \\
&\geq \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 - \theta_n (\|x_{n+1} - x_n\|^2 + \|x_n - x_{n-1}\|^2) \\
&= (1 - \theta_n)\|x_{n+1} - x_n\|^2 - \theta_n (1 - \theta_n)\|x_n - x_{n-1}\|^2.
\end{aligned} \tag{3.5}
$$

Substituting (3.4) and (3.5) into (3.3), we get

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|w_n - x^*\|^2 - \|x_{n+1} - w_n\|^2 \\
&\leq (1 + \theta_n)\|x_n - x^*\|^2 - \theta_n \|x_{n-1} - x^*\|^2 + \theta_n (1 + \theta_n)\|x_n - x_{n-1}\|^2 \\
&\quad - (1 - \theta_n)\|x_{n+1} - x_n\|^2 + \theta_n (1 - \theta_n)\|x_n - x_{n-1}\|^2 \\
&= (1 + \theta_n)\|x_n - x^*\|^2 - \theta_n \|x_{n-1} - x^*\|^2 + 2\theta_n \|x_n - x_{n-1}\|^2 \\
&\quad - (1 - \theta_n)\|x_{n+1} - x_n\|^2.
\end{aligned} \tag{3.6}
$$

Define $\Gamma_n := \|x_n - x^*\|^2 - \theta_n \|x_{n-1} - x^*\|^2 + 2\theta_n \|x_n - x_{n-1}\|^2$.
By the difinition of Γ_n and (3.6) we have

$$
\begin{aligned}
\Gamma_{n+1} - \Gamma_n &= \|x_{n+1} - x^*\|^2 - \theta_{n+1}\|x_n - x^*\|^2 + 2\theta_{n+1}\|x_{n+1} - x_n\|^2 \\
&\quad - \|x_n - x^*\|^2 + \theta_n \|x_{n-1} - x^*\|^2 - 2\theta_n \|x_n - x_{n-1}\|^2 \\
&\leq (1 + \theta_n)\|x_n - x^*\|^2 - \theta_n \|x_{n-1} - x^*\|^2 + 2\theta_n \|x_n - x_{n-1}\|^2 \\
&\quad - (1 - \theta_n)\|x_{n+1} - x_n\|^2 - \theta_{n+1}\|x_n - x^*\|^2 + 2\theta_{n+1}\|x_{n+1} - x_n\|^2 \\
&\quad - \|x_n - x^*\|^2 + \theta_n \|x_{n-1} - x^*\|^2 - 2\theta_n \|x_n - x_{n-1}\|^2 \\
&= (\theta_n - \theta_{n+1})\|x_n - x^*\|^2 + (2\theta_{n+1} + \theta_n - 1)\|x_{n+1} - x_n\|^2 \\
&\leq (3\theta_{n+1} - 1)\|x_{n+1} - x_n\|^2 \\
&\leq -\frac{1}{4}\|x_{n+1} - x_n\|^2.
\end{aligned} \tag{3.7}
$$

Hence, $\{\Gamma_n\}$ is non-increasing ($n \geq 1$). Moreover, from the definition of Γ_n, we obtain

$$
\begin{aligned}
\|x_n - x^*\|^2 &= \Gamma_n + \theta_n \|x_{n-1} - x^*\|^2 - 2\theta_n \|x_n - x_{n-1}\|^2 \\
&\leq \theta_n \|x_{n-1} - x^*\|^2 + \Gamma_n \\
&\leq \frac{1}{4}\|x_{n-1} - x^*\|^2 + \Gamma_1 \\
&\leq \frac{1}{4^2}\|x_{n-2} - x^*\|^2 + \Gamma_1 + \frac{1}{4}\Gamma_1 \\
&\quad \vdots \\
&\leq \frac{1}{4^{n-1}}\|x_1 - x^*\|^2 + (1 + \frac{1}{4} + \cdots + \frac{1}{4^{n-2}})\Gamma_1 \\
&\leq \frac{1}{4^{n-1}}\|x_1 - x^*\|^2 + \frac{4}{3}\Gamma_1.
\end{aligned} \tag{3.8}
$$

Therefore, $\{x_n\}$ is bounded. Moreover $\{w_n\}$, $\{y_n\}$, $\{Aw_n\}$ and $\{Ay_n\}$ are bounded.
By the definition of Γ_{n+1} and (3.8), we have

$$
-\Gamma_{n+1} = -\|x_{n+1} - x^*\|^2 + \theta_{n+1}\|x_n - x^*\|^2 - 2\theta_{n+1}\|x_{n+1} - x_n\|^2
$$

$$\begin{aligned}&\le \theta_{n+1}\|x_n - x^*\|^2 \\ &\le \frac{1}{4^n}\|x_1 - x^*\|^2 + \frac{1}{3}\Gamma_1. \end{aligned} \tag{3.9}$$

From (3.7) and (3.9), we get

$$\frac{1}{4}\sum_{k=1}^{n}\|x_{k+1} - x_k\|^2 \le \Gamma_1 - \Gamma_{n+1} \le \frac{1}{4^n}\|x_1 - x^*\|^2 + \frac{4}{3}\Gamma_1.$$

Therefore,

$$\sum_{n=1}^{\infty}\|x_{n+1} - x_n\|^2 \le \frac{\Gamma_1}{3} < +\infty. \tag{3.10}$$

Thus,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.11}$$

From (3.6) we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 \\ &\quad - (1-\theta_n)\|x_{n+1} - x_n\|^2 \\ &\le (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 \\ &= \|x_n - x^*\|^2 + \theta_n(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2) + 2\theta_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.12}$$

Invoking Lemma 2.7 in (3.12), we get $\lim_{n\to\infty}\|x_n - x^*\|$ exists. ∎

Lemma 3.3. *Assume that Condition 1-4 hold and $\{x_n\}$ is a sequence generated by Algorithm 1. Suppose $\lim_{n\to\infty}\|y_n - w_n\| = 0$. If v^* is one of the weak cluster points of $\{x_n\}$, then we have at least one of the following: $v^* \in S_D$ or $Av^* = 0$.*

Proof. By Lemma 3.2, $\{x_n\}$ is bounded. Hence we can let v^* be a weak cluster point of $\{x_n\}$. Then we can choose a subsequence of $\{x_n\}$, denoted by $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup v^* \in H$. We consider the following two possible cases.

Case I: Suppose that $\limsup_{k\to\infty}\|Ax_{n_k}\| = 0$. Then $\lim_{k\to\infty}\|Ax_{n_k}\| = \liminf_{k\to\infty}\|Ax_{n_k}\| = 0$. According to Condition 3, we obtain

$$Av^* = 0.$$

Case II: Suppose that $\limsup_{k\to\infty}\|Ax_{n_k}\| > 0$. Then without loss of generality, we can choose a subsequence of $\{Ax_{n_k}\}$ still denoted by $\{Ax_{n_k}\}$ such that $\lim_{k\to\infty}\|Ax_{n_k}\| = M_1 > 0$. From $y_n = P_C(w_n - \lambda_n Aw_n)$, we get

$$\langle w_n - \lambda_n Aw_n - y_n, y - y_n\rangle \le 0, \ \forall y \in C.$$

So,

$$\begin{aligned}0 &\le \langle y_{n_k} - w_{n_k} + \lambda_{n_k} A w_{n_k}, y - y_{n_k}\rangle \\ &= \langle y_{n_k} - w_{n_k}, y - y_{n_k}\rangle + \lambda_{n_k}\langle A w_{n_k}, y - y_{n_k}\rangle \\ &= \langle y_{n_k} - w_{n_k}, y - y_{n_k}\rangle + \lambda_{n_k}\langle A w_{n_k}, y - w_{n_k}\rangle + \lambda_{n_k}\langle A w_{n_k}, w_{n_k} - y_{n_k}\rangle,\ \forall y \in C.\end{aligned} \tag{3.13}$$

From $\lim\limits_{n\to\infty} \|w_n - y_n\| = 0$, we obtain

$$0 \le \liminf_{k\to\infty}\langle A w_{n_k}, y - w_{n_k}\rangle \le \limsup_{k\to\infty}\langle A w_{n_k}, y - w_{n_k}\rangle < \infty,\ \forall y \in C. \tag{3.14}$$

Based on (3.14), we consider the following two cases under case II:

Case 1: Suppose that $\limsup\limits_{k\to\infty}\langle A w_{n_k}, y - w_{n_k}\rangle > 0, \forall y \in C$. Then we can choose a subsequence of $\{w_{n_k}\}$ denoted by $\{w_{n_{k_j}}\}$ such that $\lim\limits_{j\to\infty}\langle A w_{n_{k_j}}, y - w_{n_{k_j}}\rangle > 0$. Thus, there exists $j_0 \ge 1$ such that $\langle A w_{n_{k_j}}, y - w_{n_{k_j}}\rangle > 0, \forall j \ge j_0$, by the quasimonotonicity of A on H, we have

$$\langle Ay, y - w_{n_{k_j}}\rangle \ge 0, \forall y \in C, j \ge j_0. \tag{3.15}$$

From $w_n = x_n + \theta_n(x_n - x_{n-1})$ and (3.11), we have $\lim\limits_{n\to\infty}\|w_n - x_n\| = 0$, thus, $w_{n_k} \rightharpoonup v^*$. Letting $j \to \infty$ in (3.15), we have $\langle Ay, y - v^*\rangle \ge 0, \forall y \in C$. Therefore, $v^* \in S_D$.

Case 2: Suppose that $\limsup\limits_{k\to\infty}\langle A w_{n_k}, y - w_{n_k}\rangle = 0, \forall y \in C$. Then by (3.14), we get

$$\lim_{k\to\infty}\langle A w_{n_k}, y - w_{n_k}\rangle = 0,\ \forall y \in C, \tag{3.16}$$

from which we get

$$\langle A w_{n_k}, y - w_{n_k}\rangle + |\langle A w_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1} > 0,\ \forall y \in C. \tag{3.17}$$

From $\lim\limits_{k\to\infty}\|w_{n_k} - x_{n_k}\| = 0$ and L-Lipschitz continuity of A, we have $\lim\limits_{k\to\infty}\|A w_{n_k} - A x_{n_k}\| = 0$. Thus, $\lim\limits_{k\to\infty}\|A w_{n_k}\| = \lim\limits_{k\to\infty}\|A w_{n_k} - A x_{n_k} + A x_{n_k}\| = \|\lim\limits_{k\to\infty}(A w_{n_k} - A x_{n_k} + A x_{n_k})\| = \|\lim\limits_{k\to\infty} A x_{n_k}\| = M_1 > 0$, we can find $k_0 \ge 1$ such that $\|A w_{n_k}\| \ge \frac{M_1}{2}, \forall k \ge k_0$.

We set $b_{n_k} = \frac{A w_{n_k}}{\|A w_{n_k}\|^2}$, then $\langle A w_{n_k}, b_{n_k}\rangle = 1$. Therefore, by (3.17), we get

$$\Big\langle A w_{n_k}, y + b_{n_k}\Big[|\langle A w_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big] - w_{n_k}\Big\rangle > 0,\ \forall y \in C. \tag{3.18}$$

From A is quasimonotone on H, we obtain

$$\Bigg\langle A\Big(y + b_{n_k}\Big[|\langle A w_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big]\Big), y + b_{n_k}\Big[|\langle A w_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big] - w_{n_k}\Bigg\rangle \ge 0,\ \forall y \in C.$$

This implies that

$$\Big\langle Ay, y + b_{n_k}\Big[|\langle A w_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big] - w_{n_k}\Big\rangle$$

$$\geq \Big\langle Ay - A(y + b_{n_k}\Big[|\langle Aw_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big]), y + b_{n_k}\Big[|\langle Aw_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big] - w_{n_k}\Big\rangle$$
$$\geq -\|Ay - A(y + b_{n_k}\Big[|\langle Aw_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big])\| \cdot \|y + b_{n_k}\Big[|\langle Aw_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big] - w_{n_k}\|$$
$$\geq -L\|b_{n_k}\Big[|\langle Aw_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big]\| \cdot \|y + b_{n_k}\Big[|\langle Aw_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big] - w_{n_k}\|$$
$$= \frac{-L}{\|Aw_{n_k}\|}\Big(|\langle Aw_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big) \cdot \|y + b_{n_k}\Big[|\langle Aw_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big] - w_{n_k}\|$$
$$\geq \frac{-2L}{M_1}\Big(|\langle Aw_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big)M_2, \ \forall y \in C, k \geq k_0, \tag{3.19}$$

for some $M_2 > 0$, where the existence of M_2 is from the boundedness of $\Big\{y+b_{n_k}\Big[|\langle Aw_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big] - w_{n_k}\Big\}$.
From (3.16) we have $\lim\limits_{k\to\infty}\Big(|\langle Aw_{n_k}, y - w_{n_k}\rangle| + \frac{1}{k+1}\Big) = 0, \forall y \in C$. Thus, as $k \to \infty$ in (3.38), we get $\langle Ay, y - v^*\rangle \geq 0, \forall y \in C$. Therefore, $v^* \in S_D$. □

Theorem 3.4. *Let $\{x_n\}$ be generated by Algorithm 1 such that Condition 1-4 hold and $Ax \neq 0, \forall x \in C$ (otherwise, $x \in S$). Then $\{x_n\}$ converges weakly to an element of S_D.*

Proof. We first show that $\lim\limits_{n\to\infty}\|w_n - y_n\| = 0$. We pick a point $x^* \in S_D$.
Since x^* be a solution to problem (2.1), we have

$$\langle Ay_n, x^* - y_n\rangle \leq 0. \tag{3.20}$$

From $y_n = P_C(w_n - \lambda_n Aw_n)$ and Lemma 2.5, we get

$$\langle w_n - \lambda_n Aw_n - y_n, y - y_n\rangle \leq 0, \ \forall y \in C. \tag{3.21}$$

From and (3.20) and (3.21), we can deduce

$$\begin{aligned} h_n(x^*) &= \langle w_n - y_n - \lambda_n(Aw_n - Ay_n), x^* - y_n\rangle \\ &= \langle w_n - y_n - \lambda_n Aw_n, x^* - y_n\rangle + \lambda_n\langle Ay_n, x^* - y_n\rangle \\ &\leq 0. \end{aligned}$$

Using (3.2), we have

$$\begin{aligned} h_n(w_n) &= \langle w_n - y_n - \lambda_n(Aw_n - Ay_n), w_n - y_n\rangle \\ &= \|w_n - y_n\|^2 - \lambda_n\langle Aw_n - Ay_n, w_n - y_n\rangle \\ &\geq \|w_n - y_n\|^2 - \lambda_n\|Aw_n - Ay_n\| \ \|w_n - y_n\| \\ &\geq \|w_n - y_n\|^2 - \mu q_n\frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2 \end{aligned}$$

$$= (1 - \mu q_n \frac{\lambda_n}{\lambda_{n+1}})\|w_n - y_n\|^2.$$

Since $\lim_{n\to\infty}\left(1 - \mu q_n \frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > \frac{1-\mu}{2} > 0$, there exists $n_0 \in \mathbb{N}$ such that $1 - \mu q_n \frac{\lambda_n}{\lambda_{n+1}} > \frac{1-\mu}{2}$ for all $n \geq n_0$. Therefore,

$$h_n(w_n) \geq \frac{1-\mu}{2}\|w_n - y_n\|^2, \ \forall n \geq n_0. \tag{3.22}$$

For all $u, v \in H$, we have

$$\begin{aligned}\|h_n(u) - h_n(v)\| &= \|\langle w_n - y_n - \lambda_n(Aw_n - Ay_n), u - v\rangle\| \\ &\leq \|w_n - y_n - \lambda_n(Aw_n - Ay_n)\| \cdot \|u - v\| \\ &\leq M_3\|u - v\|,\end{aligned}$$

where the existence of M_3 is from the boundedness of $\{w_n - y_n - \lambda_n(Aw_n - Ay_n)\}$. This implies that $h_n(\cdot)$ is M_3-Lipschitz continuous on H. Using Lemma 2.6, we have

$$dist(w_n, T_n) \geq \frac{1}{M_3}h_n(w_n). \tag{3.23}$$

On the other hand, from Lemma 2.4, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|P_{T_n}(w_n) - x^*\|^2 \\ &\leq \|w_n - x^*\|^2 - \|P_{T_n}(w_n) - w_n\|^2 \\ &= \|w_n - x^*\|^2 - dist^2(w_n, T_n).\end{aligned} \tag{3.24}$$

From (3.22), (3.23) and (3.24), we obtain

$$\|x_{n+1} - x^*\|^2 \leq \|w_n - x^*\|^2 - \left[\frac{1}{M_3}\frac{1-\mu}{2}\|w_n - y_n\|^2\right]^2, \ \forall n \geq n_0. \tag{3.25}$$

From $w_n = x_n + \theta_n(x_n - x_{n-1})$ and $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have $\lim_{n\to\infty}\|w_n - x_n\| = 0$. Combining $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty}\|w_n - x_n\| = 0$ and (3.25), we get

$$\begin{aligned}\left[\frac{1}{M_3}\frac{1-\mu}{2}\|w_n - y_n\|^2\right]^2 &\leq \|w_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &= (\|w_n - x^*\| + \|x_{n+1} - x^*\|)(\|w_n - x^*\| - \|x_{n+1} - x^*\|) \\ &\leq M_4\|w_n - x_{n+1}\| \\ &\leq M_4(\|w_n - x_n\| + \|w_n - x_{n+1}\|), \ \forall n \geq n_0\end{aligned} \tag{3.26}$$

where the existence of M_3 is from the boundedness of $\{\|w_n - x^*\| + \|x_{n+1} - x^*\|\}$. Letting $n \to \infty$ in (3.26), we get

$$\lim_{n\to\infty}\|w_n - y_n\| = 0.$$

By Lemma 3.2, $\{x_n\}$ is bounded, hence, let z be a weak cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$, such that $x_{n_k} \rightharpoonup z, k \to \infty$, also from $\|x_n - y_n\| \to$

$0, n \to \infty$, we get $y_{n_k} \rightharpoonup z, k \to \infty$. Since C is closed, we have that $z \in C$. Since $Ax \neq 0, \forall x \in C$, we get $Az \neq 0$. By $\lim\limits_{n\to\infty} \|w_n - y_n\| = 0$ and Lemma 3.3, we get $z \in S_D$. Therefore,
(1) by Lemma 3.2, $\lim\limits_{n\to\infty} \|x_n - z\|$ exists for any $z \in S_D$,
(2) every sequential weak cluster point of $\{x_n\}$ is in S_D.
Using Lemma 2.8, we get $\{x_n\}$ converges weakly to a point in S_D. □

3.2 Modified viscosity-type inertial projection algorithm

Condition 5 Let $\{\alpha_n\} \subset (0,1)$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$, $\{\xi_n\}$ be a positive sequence such that $\lim\limits_{n\to\infty} \frac{\xi_n}{\alpha_n} = 0$;
Condition 6 The mapping $f : H \to H$ is a contraction mapping with contraction parameter $\beta \in [0,1)$.

Algorithm 2 Modified viscosity-type inertial projection algorithm

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Iterative step:
Initialization: Take the parameters $\mu \in (0,1)$, $\theta > 0$ and $\lambda_1 > 0$. Choose $\{a_n\} \subset [0,\infty)$ such that $\sum\limits_{n=1}^{\infty} a_n < +\infty$ and $\{q_n\} \subset [1,\infty)$ such that $\lim\limits_{n\to\infty} q_n = 1$. Let $x_0, x_1 \in H$ be given starting points.
Step 1. Given the current iterates x_{n-1} and x_n. Choose $\{\theta_n\}$ such that $0 \leq \theta_n \leq \hat{\theta_n}$ with $\hat{\theta_n}$ defined by

$$\hat{\theta_n} = \begin{cases} \min\left\{\theta, \dfrac{\xi_n}{\|x_n - x_{n-1}\|}\right\}, & if x_n \neq x_{n-1}, \\ \theta, & otherwise, \end{cases} \tag{3.27}$$

compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}),$$

Step 2. Compute

$$y_n = P_C(w_n - \lambda_n A w_n).$$

where the step size λ_{n+1} is updated by

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu q_n \|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n + a_n\right\}, & if Aw_n - Ay_n \neq 0, \\ \lambda_n + a_n, & otherwise. \end{cases} \tag{3.28}$$

If $w_n = y_n$ or $Aw_n = 0$, STOP. Otherwise,
Step 3. Compute

$$u_n = P_{T_n}(w_n),$$

where

$$T_n := \{w \in H : h_n(w) \leq 0\}$$

and

$$h_n(w) := \langle w_n - y_n - \lambda_n(Aw_n - Ay_n), w - y_n\rangle.$$

Step 4. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)u_n.$$

Set $n := n+1$, and go to **Step 1**.

Remark 3.5. By Condition 5, we can easily verify the following results from (3.27):

$$\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0, \quad \lim_{n\to\infty}\theta_n\|x_n - x_{n-1}\| = 0.$$

Lemma 3.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 2. Then $\{x_n\}$ is bounded and the following inequality holds under Condition 1-6, where $x^* \in S_D$:*

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq& \Big(1 - \frac{2\alpha_n(1-\beta)}{1-\alpha_n\beta}\Big)\|x_n - x^*\|^2 + \frac{2\alpha_n(1-\beta)}{1-\alpha_n\beta}\Big\{\frac{\alpha_n Q_3}{2(1-\beta)} \\ &+ \frac{3Q_2(1-\alpha_n)^2}{2(1-\beta)}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \frac{1}{1-\beta}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle\Big\} \\ &- \frac{(1-\alpha_n)^2}{1-\alpha_n\beta}\left(\frac{1-\mu}{2M_3}\right)^2\|w_n - y_n\|^4.\end{aligned}$$

Proof. First, we show that $\{x_n\}$ is bounded. From (3.25), we have

$$\begin{aligned}\|u_n - x^*\| &\leq \|w_n - x^*\| \\ &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\| \\ &\leq \|x_n - x^*\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\leq \|x_n - x^*\| + \alpha_n Q_1, \end{aligned} \tag{3.29}$$

where the existence of Q_1 is from $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.
From (3.29) and Condition 6, we obtain

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + (1-\alpha_n)u_n - x^*\| \\ &\leq \alpha_n\|f(x_n) - x^*\| + (1-\alpha_n)\|u_n - x^*\| \\ &\leq \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\Big(\|x_n - x^*\| + \alpha_n Q_1\Big) \\ &\leq (\alpha_n\beta + 1 - \alpha_n)\|x_n - x^*\| + \alpha_n\Big(\|f(x^*) - x^*\| + (1-\alpha_n)Q_1\Big) \\ &= (1 - \alpha_n(1-\beta))\|x_n - x^*\| + \alpha_n(1-\beta)\frac{\|f(x^*) - x^*\| + Q_1}{1-\beta} \\ &\leq \max\Big\{\|x_n - x^*\|, \frac{\|f(x^*) - x^*\| + Q_1}{1-\beta}\Big\} \\ &\ \ \vdots\end{aligned}$$

$$\leq \max\Big\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\| + Q_1}{1-\beta}\Big\}.$$

This implies that $\{x_n\}$ is bounded.
Using Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\|w_n - x^*\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 + 2\theta_n\|x_n - x^*\| \, \|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - x^*\|^2 + 3Q_2\theta_n\|x_n - x_{n-1}\|, \end{aligned} \tag{3.30}$$

where $Q_2 := \sup\limits_{n\in\mathbb{N}}\{\|x_n - x^*\|, \theta_n\|x_n - x_{n-1}\|\}$.
Applying Lemma 2.3, (3.25) and (3.30), we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)u_n - x^*\|^2 \\ &= \|(1-\alpha_n)(u_n - x^*) + \alpha_n(f(x_n) - x^*)\|^2 \\ &\leq (1-\alpha_n)^2\|u_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\alpha_n)^2\Big\{\|w_n - x^*\|^2 - \Big[\frac{1}{M_3}\frac{1-\mu}{2}\|w_n - y_n\|^2\Big]^2\Big\} \\ &\quad + 2\alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\alpha_n)^2\Big[\|x_n - x^*\|^2 + 3Q_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| - \Big(\frac{1-\mu}{2M_3}\Big)^2\|w_n - y_n\|^4\Big] \\ &\quad + 2\alpha_n\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\alpha_n)^2\Big[\|x_n - x^*\|^2 + 3Q_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| - \Big(\frac{1-\mu}{2M_3}\Big)^2\|w_n - y_n\|^4\Big] \\ &\quad + 2\alpha_n\beta\|x_n - x^*\| \, \|x_{n+1} - x^*\| + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\alpha_n)^2\Big[\|x_n - x^*\|^2 + 3Q_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| - \Big(\frac{1-\mu}{2M_3}\Big)^2\|w_n - y_n\|^4\Big] \\ &\quad + \alpha_n\beta(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &= \Big((1-\alpha_n)^2 + \alpha_n\beta\Big)\|x_n - x^*\|^2 + \alpha_n\beta\|x_{n+1} - x^*\|^2 \\ &\quad + 3Q_2\alpha_n\frac{\theta_n}{\alpha_n}(1-\alpha_n)^2\|x_n - x_{n-1}\| - (1-\alpha_n)^2\Big(\frac{1-\mu}{2M_3}\Big)^2\|w_n - y_n\|^4 \\ &\quad + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle.\end{aligned}$$

From this, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \frac{1 - 2\alpha_n + \alpha_n^2 + \alpha_n\beta}{1-\alpha_n\beta}\|x_n - x^*\|^2 + \frac{3Q_2(1-\alpha_n)^2}{1-\alpha_n\beta}\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\beta}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle - \frac{(1-\alpha_n)^2}{1-\alpha_n\beta}\Big(\frac{1-\mu}{2M_3}\Big)^2\|w_n - y_n\|^4 \\ &= \frac{1 - 2\alpha_n + \alpha_n\beta}{1-\alpha_n\beta}\|x_n - x^*\|^2 + \frac{\alpha_n^2}{1-\alpha_n\beta}\|x_n - x^*\|^2 \\ &\quad + \frac{3Q_2(1-\alpha_n)^2}{1-\alpha_n\beta}\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \frac{2\alpha_n}{1-\alpha_n\beta}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle\end{aligned}$$

$$
\begin{aligned}
&- \frac{(1-\alpha_n)^2}{1-\alpha_n\beta}\Big(\frac{1-\mu}{2M_3}\Big)^2\|w_n-y_n\|^4 \\
&\le \Big(1-\frac{1-2\alpha_n(1-\beta)}{1-\alpha_n\beta}\Big)\|x_n-x^*\|^2+\frac{2\alpha_n(1-\beta)}{1-\alpha_n\beta}\Big\{\frac{\alpha_n Q_3}{2(1-\beta)} \\
&+ \frac{3Q_2(1-\alpha_n)^2}{2(1-\beta)}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\frac{1}{1-\beta}\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\Big\} \\
&- \frac{(1-\alpha_n)^2}{1-\alpha_n\beta}\Big(\frac{1-\mu}{2M_3}\Big)^2\|w_n-y_n\|^4, \qquad (3.31)
\end{aligned}
$$

where $Q_3 := \sup\{\|x_n - x^*\|^2 : n \in \mathbb{N}\}$. This completes the proof. □

Lemma 3.7. *Assume that $\{w_n\}$ and $\{y_n\}$ are generated by Algorithm 2 and Condition 1-6 hold. Suppose that there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\{w_{n_k}\} \rightharpoonup v^* \in H$ and $\lim_{k\to\infty}\|y_{n_k} - w_{n_k}\| = 0$, then we have at least one of the following: $v^* \in S_D$ or $Av^* = 0$.*

Proof. By Lemma 3.6, we know that $\{x_n\}$ is bounded, then $\{w_n\}$ is bounded. Hence we can let v^* be a weak cluster point of $\{w_n\}$. Then we can choose a subsequence of $\{w_n\}$, denoted by $\{w_{n_k}\}$ such that $w_{n_k} \rightharpoonup v^* \in H$.
We consider the following two possible cases.
Case I: Suppose that $\limsup_{k\to\infty}\|Aw_{n_k}\| = 0$. Then $\lim_{k\to\infty}\|Aw_{n_k}\| = \liminf_{k\to\infty}\|Aw_{n_k}\| = 0$. According to Condition 3, we obtain

$$Av^* = 0.$$

Case II: Suppose that $\limsup_{k\to\infty}\|Aw_{n_k}\| > 0$. Then without loss of generality, we can choose a subsequence of $\{Aw_{n_k}\}$ still denoted by $\{Aw_{n_k}\}$ such that $\lim_{k\to\infty}\|Aw_{n_k}\| = P_1 > 0$.
From $y_n = P_C(w_n - \lambda_n Aw_n)$, we get

$$\langle w_n - \lambda_n Aw_n - y_n, y - y_n\rangle \le 0,\ \forall y \in C.$$

So,

$$
\begin{aligned}
0 &\le \langle y_{n_k} - w_{n_k} + \lambda_{n_k}Aw_{n_k}, y - y_{n_k}\rangle \\
&= \langle y_{n_k} - w_{n_k}, y - y_{n_k}\rangle + \lambda_{n_k}\langle Aw_{n_k}, y - y_{n_k}\rangle \\
&= \langle y_{n_k} - w_{n_k}, y - y_{n_k}\rangle + \lambda_{n_k}\langle Aw_{n_k}, y - w_{n_k}\rangle + \lambda_{n_k}\langle Aw_{n_k}, w_{n_k} - y_{n_k}\rangle,\ \forall y \in C.
\end{aligned} \tag{3.32}
$$

From $\lim_{n\to\infty}\|w_{n_k} - y_{n_k}\| = 0$, we obtain

$$0 \le \liminf_{k\to\infty}\langle Aw_{n_k}, y - w_{n_k}\rangle \le \limsup_{k\to\infty}\langle Aw_{n_k}, y - w_{n_k}\rangle < \infty,\ \forall y \in C. \tag{3.33}$$

Based on (3.33), we consider the following two cases under case II:
Case 1: Suppose that $\limsup_{k\to\infty}\langle Aw_{n_k}, y - w_{n_k}\rangle > 0, \forall y \in C$. Then we can choose a subsequence of $\{w_{n_k}\}$ denoted by $\{w_{n_{k_j}}\}$ such that $\lim_{j\to\infty}\langle Aw_{n_{k_j}}, y - w_{n_{k_j}}\rangle > 0$. Thus, there

exists $j_1 \geq 1$ such that $\langle Aw_{n_{k_j}}, y - w_{n_{k_j}} \rangle > 0, \forall j \geq j_1$, by the quasimonotonicity of A on H, we have

$$\langle Ay, y - w_{n_{k_j}} \rangle \geq 0, \forall y \in C, j \geq j_1. \tag{3.34}$$

Letting $j \to \infty$ in (3.34), we have $\langle Ay, y - v^* \rangle \geq 0, \forall y \in C$. Therefore, $v^* \in S_D$.
Case 2: Suppose that $\limsup_{k\to\infty} \langle Aw_{n_k}, y - w_{n_k} \rangle = 0, \forall y \in C$. Then by (3.33), we get

$$\lim_{k\to\infty} \langle Aw_{n_k}, y - w_{n_k} \rangle = 0, \ \forall y \in C, \tag{3.35}$$

from which we get

$$\langle Aw_{n_k}, y - w_{n_k} \rangle + |\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} > 0, \ \forall y \in C. \tag{3.36}$$

From $\lim_{k\to\infty} \|Aw_{n_k}\| = P_1 > 0$, we can find $k_1 \geq 1$ such that $\|Aw_{n_k}\| \geq \frac{P_1}{2}, \forall k \geq k_1$.
We set $b_{n_k} = \frac{Aw_{n_k}}{\|Aw_{n_k}\|^2}$, then $\langle Aw_{n_k}, b_{n_k} \rangle = 1$. Therefore, by (3.36), we get

$$\left\langle Aw_{n_k}, y + b_{n_k} \left[|\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} \right] - w_{n_k} \right\rangle > 0, \ \forall y \in C. \tag{3.37}$$

From the quasimonotonicity of A, we obtain for all $y \in C$,

$$\left\langle A\Big(y + b_{n_k} \left[|\langle Aw_{n_k}, y - y_{n_k} \rangle| + \frac{1}{k+1} \right] \Big), y + b_{n_k} \left[|\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} \right] - w_{n_k} \right\rangle \geq 0.$$

This implies that

$$\begin{aligned}
&\left\langle Ay, y + b_{n_k} \left[|\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} \right] - w_{n_k} \right\rangle \\
&\geq \left\langle Ay - A(y + b_{n_k} \left[|\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} \right]), y + b_{n_k} \left[|\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} \right] \right. \\
&\quad \left. - w_{n_k} \right\rangle \\
&\geq -\|Ay - A(y + b_{n_k} \left[|\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} \right])\| \cdot \|y + b_{n_k} \Big[|\langle Aw_{n_k}, y - w_{n_k} \rangle| \\
&\quad + \frac{1}{k+1} \Big] - w_{n_k}\| \\
&\geq -L\|b_{n_k} \left[|\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} \right]\| \cdot \|y + b_{n_k} \left[|\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} \right] - w_{n_k}\| \\
&= \frac{-L}{\|Aw_{n_k}\|} \Big(|\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} \Big) \cdot \|y + b_{n_k} \left[|\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} \right] - w_{n_k}\| \\
&\geq \frac{-2L}{P_1} \Big(|\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} \Big) P_2, \ \forall y \in C, k \geq k_0,
\end{aligned} \tag{3.38}$$

for some $P_2 > 0$, where the existence of P_2 is from the boundedness of $\Big\{ y + b_{n_k} \Big[|\langle Aw_{n_k}, y - w_{n_k} \rangle| + \frac{1}{k+1} \Big] - w_{n_k} \Big\}$.

From (3.35) we have $\lim_{k\to\infty}\left(|\langle Aw_{n_k}, y-w_{n_k}\rangle| + \frac{1}{k+1}\right) = 0, \forall y \in C$. Thus, as $k \to \infty$ in (3.38), we get $\langle Ay, y-v^*\rangle \geq 0, \forall y \in C$. Therefore, $v^* \in S_D$. ∎

Theorem 3.8. *Let $\{x_n\}$ be a sequence generated by Algorithm 2 such that Condition 1-6 hold and $Ax \neq 0, \forall x \in C$. Then, $\{x_n\}$ converges strongly to an element $p \in S_D \subset S$, where $p = P_{S_D} \circ f(p)$.*

Proof. From Lemma 3.6, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 \leq &\Big(1-\frac{1-2\alpha_n(1-\beta)}{1-\alpha_n\beta}\Big)\|x_n-p\|^2 + \frac{2\alpha_n(1-\beta)}{1-\alpha_n\beta}\Big\{\frac{\alpha_n Q_3}{2(1-\beta)} \\ &+\frac{3Q_2(1-\alpha_n)^2}{2(1-\beta)}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + \frac{1}{1-\beta}\langle f(p)-p, x_{n+1}-p\rangle\Big\}. \end{aligned} \tag{3.39}$$

Next, we claim that the sequence $\{\|x_n-p\|\}$ converges to zero. Applying Lemma 2.9 to (3.39), we know that it is sufficient to show $\limsup_{k\to\infty}\langle f(p)-p, x_{n_k+1}-p\rangle \leq 0$ for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ satisfying

$$\liminf_{k\to\infty}(\|x_{n_k+1}-p\| - \|x_{n_k}-p\|) \geq 0. \tag{3.40}$$

From Lemma 3.6, we obtain

$$\begin{aligned} &\frac{(1-\alpha_{n_k})^2}{1-\alpha_{n_k}\beta}\Big(\frac{1-\mu}{2M_3}\Big)^2\|w_{n_k}-y_{n_k}\|^4 \\ &\leq \Big(1-\frac{2\alpha_{n_k}(1-\beta)}{1-\alpha_{n_k}\beta}\Big)\|x_{n_k}-p\|^2 - \|x_{n_k+1}-p\|^2 \\ &\quad + \frac{2\alpha_{n_k}(1-\beta)}{1-\alpha_{n_k}\beta}\Big\{\frac{\alpha_{n_k}Q_3}{2(1-\beta)} + \frac{3Q_2(1-\alpha_{n_k})^2}{2(1-\beta)}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\| \\ &\quad + \frac{1}{1-\beta}\langle f(p)-p, x_{n_k+1}-p\rangle\Big\}. \end{aligned} \tag{3.41}$$

Letting $k\to\infty$ in (3.41), applying $\lim_{k\to\infty}\alpha_{n_k} = 0$ and (3.40), we obtain

$$\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\| = 0. \tag{3.42}$$

Using the definition of w_n and Remark 3.5 we have

$$\lim_{k\to\infty}\|w_{n_k}-x_{n_k}\| = 0. \tag{3.43}$$

Combining (3.42) and (3.43) we have

$$\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\| = 0. \tag{3.44}$$

Using Lemma 2.4 we get

$$\|u_{n_k}-p\|^2 = \|P_{T_{n_k}}(w_{n_k})-p\|^2 \leq \|w_{n_k}-p\|^2 - \|u_{n_k}-w_{n_k}\|^2. \tag{3.45}$$

On the other hand,

$$\begin{aligned}
\|w_{n_k} - p\| &= \|x_{n_k} + \theta_{n_k}(x_{n_k} - x_{n_k-1}) - p\| \\
&\le \|x_{n_k} - p\| + \alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\| \\
&\le \|x_{n_k} - p\| + \alpha_{n_k}Q_4,
\end{aligned}$$

where $Q_4 = \sup\limits_{k\in\mathbb{N}}\left\{\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\|\right\}$.
Thus, we have

$$\begin{aligned}
\|u_{n_k} - p\|^2 &\le \|w_{n_k} - p\|^2 - \|u_{n_k} - w_{n_k}\|^2 \\
&\le (\|x_{n_k} - p\| + \alpha_{n_k}Q_4)^2 - \|u_{n_k} - w_{n_k}\|^2 \\
&\le \|x_{n_k} - p\|^2 + \alpha_{n_k}(\alpha_{n_k}Q_4^2 + 2\|x_{n_k} - p\|Q_4) - \|u_{n_k} - w_{n_k}\|^2 \\
&\le \|x_{n_k} - p\|^2 - \|u_{n_k} - w_{n_k}\|^2 + \alpha_{n_k}Q_5,
\end{aligned} \tag{3.46}$$

where $Q_5 = \sup\limits_{k\in\mathbb{N}}\left\{\alpha_{n_k}Q_4^2 + 2\|x_{n_k} - p\|Q_4\right\}$.
Using Condition 6 and (3.46), we get

$$\begin{aligned}
\|x_{n_k+1} - p\|^2 &= \|\alpha_{n_k}f(x_{n_k}) + (1-\alpha_{n_k})u_{n_k} - p\|^2 \\
&= \|\alpha_{n_k}(f(x_{n_k}) - f(p) + f(p) - p) + (1-\alpha_{n_k})(u_{n_k} - p)\|^2 \\
&\le \alpha_{n_k}\|f(x_{n_k}) - f(p) + f(p) - p\|^2 + (1-\alpha_{n_k})\|u_{n_k} - p\|^2 \\
&\le \alpha_{n_k}\Big(\|f(x_{n_k}) - f(p)\| + \|f(p) - p\|\Big)^2 + (1-\alpha_{n_k})\|u_{n_k} - p\|^2 \\
&\le \alpha_{n_k}\Big(\|x_{n_k} - p\| + \|f(p) - p\|\Big)^2 + (1-\alpha_{n_k})\|u_{n_k} - p\|^2 \\
&\le \alpha_{n_k}\|x_{n_k} - p\|^2 + 2\alpha_{n_k}\|x_{n_k} - p\|\ \|f(p) - p\| + \alpha_{n_k}\|f(p) - p\|^2 \\
&\quad + (1-\alpha_{n_k})\Big(\|x_{n_k} - p\|^2 - \|u_{n_k} - w_{n_k}\|^2 + \alpha_{n_k}Q_5\Big) \\
&\le \|x_{n_k} - p\|^2 + \alpha_{n_k}Q_6 - (1-\alpha_{n_k})\|u_{n_k} - w_{n_k}\|^2,
\end{aligned}$$

where $Q_5 = \sup\limits_{k\in\mathbb{N}}\left\{2\|x_{n_k} - p\|\ \|f(p) - p\| + \|f(p) - p\|^2 + (1-\alpha_{n_k})Q_5\right\}$.
From this we have

$$(1-\alpha_{n_k})\|u_{n_k} - w_{n_k}\|^2 \le \|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2 + \alpha_{n_k}Q_6. \tag{3.47}$$

Combining (3.40) and (3.47), we get

$$\lim_{k\to\infty}\|u_{n_k} - w_{n_k}\| = 0. \tag{3.48}$$

Consequently,

$$\|x_{n_k+1} - x_{n_k}\| \le \alpha_{n_k}\|f(x_{n_k}) - x_{n_k}\| + (1-\alpha_{n_k})\|u_{n_k} - x_{n_k}\|$$

$$\le \alpha_{n_k}\|f(x_{n_k}) - x_{n_k}\| + (1-\alpha_{n_k})\Big(\|u_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\|\Big) \to 0. \tag{3.49}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\} \rightharpoonup z$ and

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \lim_{j\to\infty}\langle f(p) - p, x_{n_{k_j}} - p\rangle = \langle f(p) - p, z - p\rangle. \tag{3.50}$$

By $\lim_{n\to\infty}\|w_{n_k} - x_{n_k}\| = 0$, we have $\{w_{n_{k_j}}\} \rightharpoonup z$. Moreover, from $\lim_{j\to\infty}\|w_{n_{k_j}} - y_{n_{k_j}}\| = 0$ and Lemma 3.7 we get $z \in S_D$.
From $p = P_{S_D} \circ f(p)$, (3.50) and Lemma (2.5), we obtain

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \langle f(p) - p, z - p\rangle \le 0. \tag{3.51}$$

Combining (3.49) and (3.51), we get

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k+1} - p\rangle = \limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \langle f(p) - p, z - p\rangle \le 0. \tag{3.52}$$

Applying Lemma 2.9 to (3.39), we deduce that $\{x_n\}$ converges strongly to p. $\square$

4 Numerical experiments

In this section, we provide two numerical experiments to compare our proposed algorithms with some existing related algorithms. All the codes were written in MATLAB R2022b and performed on a PC Desktop Intel(R) Core(TM) i5-12500H @ 2.50 GHz, RAM 16.0 GB.

In all these examples, we present numerical comparisons of our proposed Algorithm 1 and Algorithm 2 with Algorithm 3.1 of Thong et al. in [28], Algorithm 1 of Shehu et al. in [22], Algorithm 3.1 of Yang in [32].

Example 4.1. Consider an operator $A : \mathbb{R}^m \to \mathbb{R}^m$ in the form of $A(x) = Mx + q$ [10], where

$$M = NN^T + G + D$$

$N \in \mathbb{R}^{m\times m}$, $G \in \mathbb{R}^{m\times m}$ is a skew-symmetric, and $D \in \mathbb{R}^{m\times m}$ is a diagonal matrix, whose diagonal entries are nonnegative (so M is positive definite), q is a vector in $\mathbb{R}^m$. The feasible set is

$$C = \{x = (x_1, \cdots, x_m) \in \mathbb{R}^m : x_i \ge -1, i = 1, \cdots m\}.$$

It is clear that A is monotone and Lipschitz continuous with Lipschitz constant $L = \|M\|$.

For experiments, all the entries of N, G are generated randomly and uniformly in $[-2, 2]$, the diagonal entries of D are in $(0, 2)$ and q is equal to the zero vector. It is easy to see that the solution of the problem in this case is $x^* = 0$.

The starting values $x_0 = x_1 = ones(m, 1)$, other parameters of our proposed algorithms and the compared algorithms are set as follows:

- Yang 2021 $\mu = 0.5$, $\mu_0 = 0.6$, $\lambda_1 = 0.5/L$, $\alpha_n = 0.1$;

- Shehu et al. 2022 $\mu = 0.5$, $\lambda_1 = 0.5/L$, $\theta_n = 0.8$, $\alpha_n = 0.2$;
- Thong et al. $\mu = 0.5$, $\lambda_0 = 0.5/L$, $\alpha_n = 1/(n+1)^2$;
- Our Alg. 1 $\mu = 0.5$, $\lambda_1 = 0.5/L$, $\theta_n = 0.8$, $a_n = 1/(n+1)^2$, $q_n = (n+1)/n$;
- Our Alg. 2 $\mu = 0.5$, $\lambda_1 = 0.5/L$, $\theta_n = 0.8$, $\alpha_n = 1/(10n+1)$, $\xi_n = 100/(n+1)^2$, $a_n = 1/(n+1)^2$, $q_n = (n+1)/n$.

The maximum number of iterations 1000 serve as a common stopping condition for all methods. At the nth step, we utilize $D_n := \|x_n - x^*\|$ to calculate the iteration error. First, we test the effect of different parameters a_n and q_n on the proposed methods with different dimensions, as shown in Figures 1-4. Next, Figures 5–8 and Table 1 show the results of the proposed methods compared to some related ones in different dimensions.

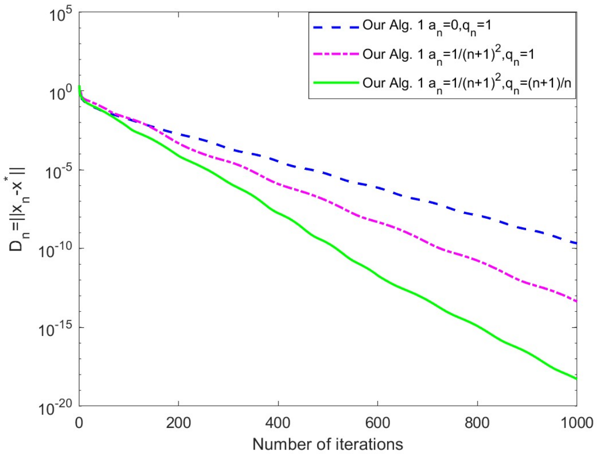

Figure 1 The behaviour of our Algorithm 1 for different a_n and q_n in Example 4.1(m=20).

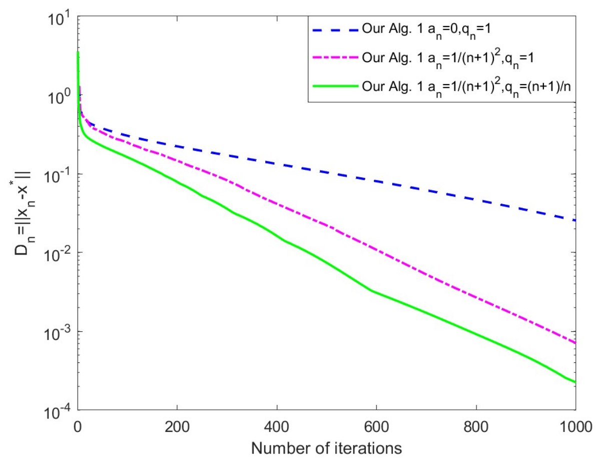

Figure 2 The behaviour of our Algorithm 1 for different a_n and q_n in Example 4.1(m=50).

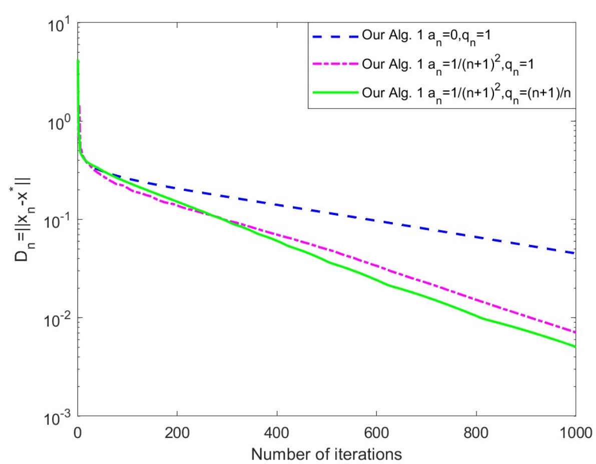

Figure 3 The behaviour of our Algorithm 1 for different a_n and q_n in Example 4.1(m=70).

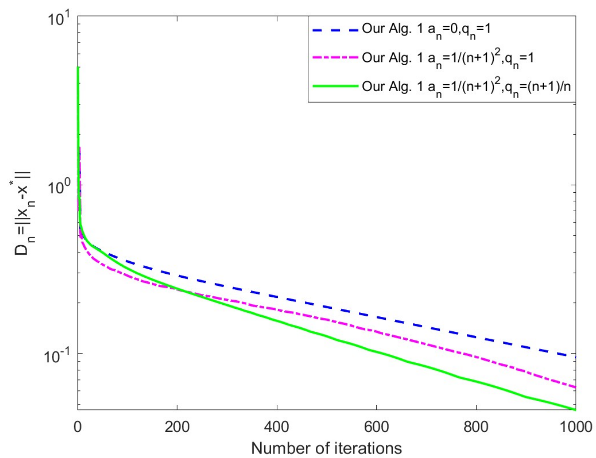

Figure 4 The behaviour of our Algorithm 1 for different a_n and q_n in Example 4.1(m=100).

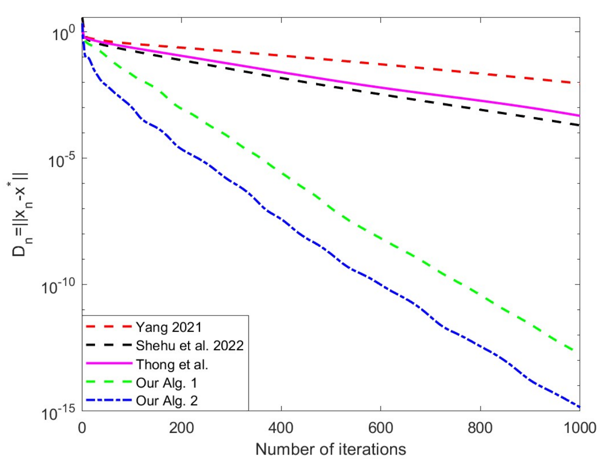

Figure 5 $m = 20$ for Example 4.1

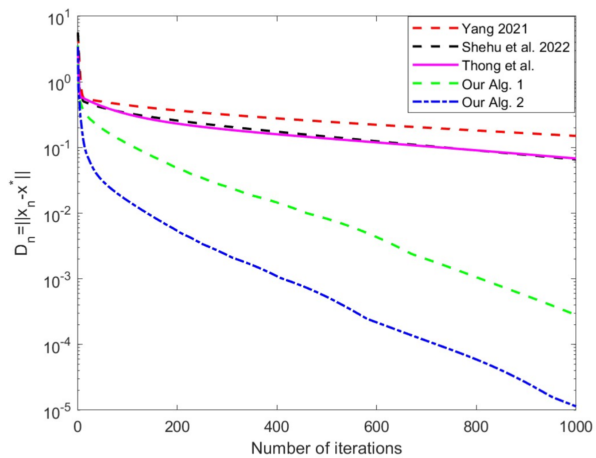

Figure 6 $m = 50$ for Example 4.1

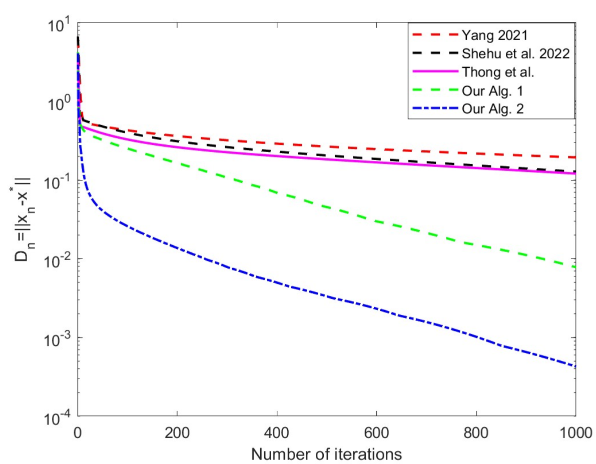

Figure 7 $m = 70$ for Example 4.1

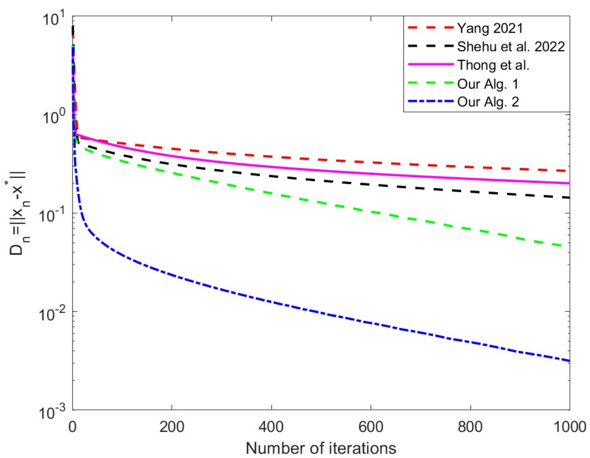

Figure 8 $m = 100$ for Example 4.1

Table 1 Numerical results for all algorithms under different dimensions in Example 4.1

	m=20		m=50		m=70		m=100	
Algorithms	Dn	CPU	Dn	CPU	Dn	CPU	Dn	CPU
Yang 2021.	0.0090	0.0065	0.1501	0.0038	0.1942	0.0068	0.2681	0.0084
Shehu et al. 2022	1.9832e-04	0.0053	0.0667	0.0031	0.1208	0.0046	0.1435	0.0056
Thong et al.	4.7412e-04	0.0051	0.0680	0.0030	0.1208	0.0048	0.2009	0.0054
Our Alg. 1	1.7966e-13	0.0022	2.7937e-04	0.0030	0.078	0.0043	0.0452	0.0058
Our Alg. 2	1.4033e-15	0.0021	1.1364e-05	0.0029	4.2681e-04	0.0043	0.0032	0.0057

Example 4.2. Consider the problem VIP whenever H is the classical $L^2[0,1]$ space with the inner product and norm given by

$$\langle x, y\rangle = \int_0^1 x(t)y(t)\mathrm{d}t,\ \|x\| = \left(\int_0^1 |x(t)|^2\ \mathrm{d}t\right)^{1/2}, \quad \forall x, y \in H.$$

Consider an operator $A : H \to H$ is given by

$$(Ax)(t) = \max\{0, x(t)\},\ t \in [0,1]\ \forall x \in H.$$

It should be noted that the operator A in the example above is 1-Lipschitz continuous and monotone on H. Let $C := \{x \in H : \|x\| \leq 1\}$ be the unit ball. The solution of the variational inequality is $x^*(t) = 0$. It is known that

$$P_C(x) = \begin{cases} \dfrac{x}{\|x\|}, & \|x\| \geq 1, \\ x, & \|x\| \leq 1. \end{cases}$$

The parameters of our proposed algorithms and the compared algorithms are set as follows:
- Yang 2021 $\mu = 0.5$, $\mu_0 = 0.6$, $\lambda_1 = 0.5$, $\alpha_n = 0.1$;
- Shehu et al. 2022 $\mu = 0.5$, $\lambda_1 = 0.5$, $\theta_n = 0.8$, $\alpha_n = 0.2$;
- Thong et al. $\mu = 0.5$, $\lambda_0 = 0.5$, $\alpha_n = 1/(n+1)^2$;
- Our Alg. 1 $\mu = 0.5$, $\lambda_1 = 0.5$, $\theta_n = 0.8$, $a_n = 1/(n+1)^2$, $q_n = (n+1)/n$;
- Our Alg. 2 $\mu = 0.5$, $\lambda_1 = 0.5$, $\theta_n = 0.8$, $\alpha_n = 1/(10n+1)$, $\xi_n = 100/(n+1)^2$, $a_n = 1/(n+1)^2$, $q_n = (n+1)/n$.

We choose the stopping criterion as error $D_n = \|x_n(t) - x^*(t)\| \leq 10^{-5}$. All the integrals are computed by the trapezoidal formula. The numerical results are described in Figure 9-11 and Table 2.

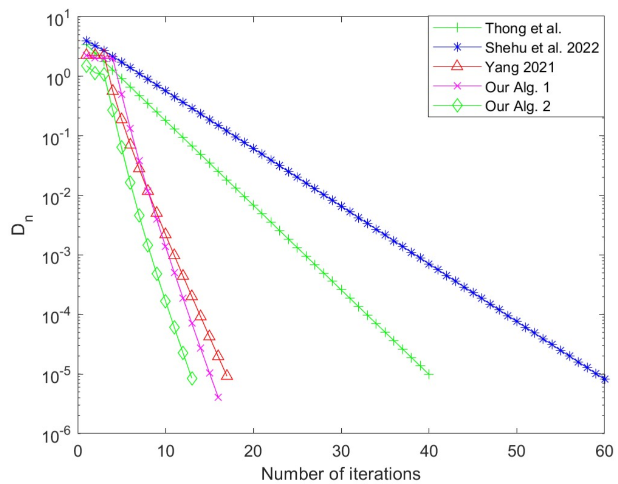

Figure 9 $x_0(t) = x_1(t) = t^2$ for Example 4.2

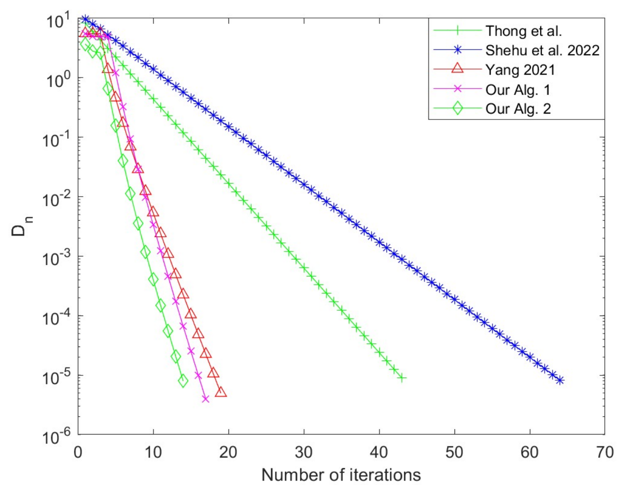

Figure 10 $x_0(t) = x_1(t) = t + \sin t$ for Example 4.2

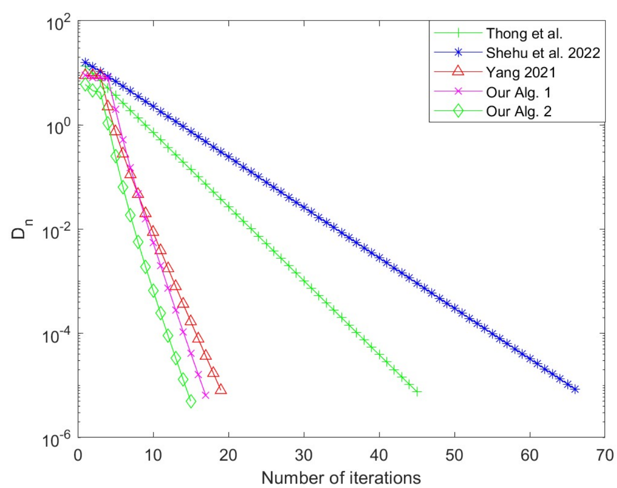

Figure 11 $x_0(t) = x_1(t) = \exp(t)$ for Example 4.2

Table 2 Numerical results for all algorithms at different initial values in Example 4.2

	$x_0(t) = x_1(t) = t^2$		$x_0(t) = x_1(t) = t + \sin t$		$x_0(t) = x_1(t) = \exp(t)$	
Algorithms	Iter.	CPU	Iter.	CPU	Iter.	CPU
Yang 2021	40	0.0072	43	0.058	45	0.0084
Shehu et al. 2022	60	0.0076	64	0.0067	66	0.0086
Thong et al.	17	0.0053	19	0.0046	19	0.0059
Our Alg. 1	16	0.0038	17	0.0041	17	0.0031
Our Alg. 2	13	0.0035	14	0.0038	15	0.0030

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